On Relations between Gluons and Gravitons

D I S S E R T A T I O N tratata

zur Erlangung des akademischen Grades

D O C T O R   R E R U M   N A T U R A L I U M

(Dr. rer. nat.)

im Fach: Physik

Spezialisierung: Theoretische Physik

eingereicht an der

Mathematisch-Naturwissenschaftlichen Fakultät

der Humboldt-Universität zu Berlin

von

Herrn M.Sc. Wadim Wormsbecher

Präsident der Humboldt-Universität zu Berlin

Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät

Prof. Dr. Elmar Kulke

Gutachter/innen:

1. Prof. Dr. Jan Plefka
2. Prof. Dr. Dirk Kreimer
3. Dr. Donal O'Connell

Tag der mündlichen Prüfung: 15.10.2019

To my beloved wife.

Lifetime brotherhood.

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018 angegebenen Hilfsmittel angefertigt habe.

Datum : ————————————-

Unterschrift : ————————————-

Diese Arbeit wurde in LaTeX unter Verwendung von TeXstudio 2.12.16 gesetzt. Für die Erstellung der Grafiken wurde das LaTeX Paket feynMF/feynMP verwendet. Zur Unterstützung einiger Rechnungen wurde Mathematica, insbesondere die Pakete S@M und EFTofPNG, herangezogen. In der Erstellung der Bibliographie wurde BiBTeX verwendet.

Zusammenfassung

In dieser Dissertation behandeln wir einige Spezialfälle von faszinierenden Beziehungen zwischen Eich- und Gravitationstheorien, welche unter dem Namen Zweifachkopie bekannt sind. Wir setzen den Schwerpunkt auf Baumlevelstreuamplituden in Einstein-(Skalar-)-Chromo-Dynamik, welche Streuungen zwischen Gluonen, massiven fundamentalen Quarks (Skalaren) und Gravitonen beschreibt. In diesen Theorien untersuchen wir den endlichen Anteil von reiner Gluonenstreuung mit zwei kollinearen Gluonen. Basierend auf einem Vorschlag von S. Stieberger und T. Taylor, stehen diese, über eine spezifische Linearkombination, in Beziehung zu Steuamplituden in Einstein-Yang-Mills Theorien, in welchen die kollinearen Gluonen durch ein Graviton ersetzt werden. Wir führen einen Beweis dieser Beziehungen unter der Ausnutzung von einer neuen Darstellung von bosonischen Baumlevelstreuamplituden durch lokalisierte Integrale auf einer Riemannschen Zahlenkugel, dem sogenannten Cachazo-He-Yuan Formalismus. Parallel dazu werden wir einen Einblick in mysteriöse Wechselwirkunen dieser Beziehungen mit Eichinvarianzverletzungen des endlichen Anteils des kollinearen Gluon Grenzwertes von Yang-Mills Streuampliuden geben. Danach behandeln wir eine andere Art von linearen Beziehungen zwischen Streuamplituden in Yang-Mills Theorie und Einstein-Yang-Mills Theorie, welche ebenfalls von S. Stieberger und T. Taylor vorgeschlagen wurden und direkt einzelne Gluonen mit einzelnen Gravitonen verbinden. Wir werden die Universalität dieser Beziehungen, in Anwesenheit von fundamental geladenen und massiven Fermionen und Skalaren beweisen. Dafür werden wir einen feynmanndiagrammatischen Zugang wählen, welcher uns eine neue Art von farbkinematischen Ersetzungsregeln liefern wird. Diese bilden Gluonen auf Gravitonen ab. Schliesslich werden wir eine neue Zweifachkopiebeziehung zwischen klassisch effektiven Wirkungen formulieren. Die effektive Wirkung eines Systems von farblich geladenen, massiven und klassischen Weltlinien, welche über Yang-Mills wechselwirken, wird mit einem System von dilatonisch geladenen, massiven und klassischen Weltlinien, welche über Dilatongravitation wechselwirken, in Verbindung gesetzt. Somit werden wir eine, aus dem Kontext von Lösungen zu störungstheoretischen Bewegungsgleichungen, sowohl für das Gluon als auch für das Graviton, derselben Systeme, bekannte Zweifachkopievorschrift, formuliert von W. Goldberger und A. Ridgway, ausbauen und verbessern.

Abstract

In this thesis we analyze several cases of mysterious connections between gauge and gravity theories, known as double copy relations. We focus on tree level scattering amplitudes in Einstein-(scalar-)-chromo-dynamics, i.e. scattering scenarios between gluons, massive fundamental quarks (scalars) and gravitons. In these scenarios we study the sub leading contribution to the adjacent collinear gluon limits in pure Yang-Mills amplitudes. Recently, S. Stieberger and T. Taylor have proposed a linear combination of amplitudes with a pair of collinear gluons to an Einstein-Yang-Mills amplitude. We present a proof of such relations using a novel representation of bosonic tree level amplitudes based on a localized integral on the Riemann sphere, called the Cachazo-He-Yuan formalism. Moreover, we give insight into an intriguing interplay between those relations and surprising gauge invariance violations of the sub-leading collinear gluon limit of Yang-Mills amplitudes. Next, we will focus on yet another set of relations between Yang-Mills amplitudes and Einstein-Yang-Mills amplitudes that were also proposed by S. Stieberger and T. Taylor. They directly relate single gluons to single gravitons. We show universality of such relations, i.e. their validity in the presence of massive fundamental quarks and scalars. For that purpose, we will use a Feynman diagrammatic approach which results in a novel color-to-kinematics rule, mapping gluons to gravitons in these scattering scenarios. Finally, we establish a novel double copy connection between classical effective actions of two massive classical worldlines which are colored and interacting in Yang-Mills theory and dilaton charged and interacting through dilaton-gravity. Doing so, we extend and improve existing work relating the same system of worldlines through a double copy at the level of perturbative solutions to the involved equations of motion for the gluon and graviton fields, as has been proposed by W. Goldberger and A. Ridgway.

Acknowledgments

I want to thank my supervisor Jan Plefka for his guidance, time, effort, support and generosity. The things you taught me will stay with me forever.
I also want to thank all the researchers, students and administrators with which I came into contact. Every single one has left an impact in my heart.
Special thanks to all my colleagues at the Humboldt-University. Thanks to you I have never felt alone at work.

Contents

Chapter 1

Introduction

Our current understanding of nature is based on two pillars. At the quantum level we have the famous standard model (SM) of elementary particles, a Yang-Mills (YM) gauge theory [1]. At the macroscopic level we have the theory of general relativity (GR). The SM describes the interaction between three generations of fermionic matter, consisting of quarks and leptons, through four gauge bosons. It explains three out of four known fundamental forces: The weak force, the strong force and electromagnetism. The Higgs boson, resulting from the Higgs mechanism based on spontaneous symmetry breaking [2], completes this theory by giving mass to the SM particles. GR explains the movement of massive bodies along geodesics. They are fully determined by the underlying geometry of the curved spacetime. Both are extremely successful theories in describing nature in their respective realm of applicability. This has been shown by a large number of experiments. One prominent example is the remarkable direct detection of gravitational waves by the Laser Interferometer Gravitational Observatory (LIGO) collaboration [3]. Another is the discovery of the Higgs boson by the ATLAS and CMS collaborations in the Large Hadron Collider (LHC) at CERN [4, 5]. Both measurements have been awarded with the Nobel prize and they validate GR and the SM to a very high degree of precision.

However, despite their success it is well known that both theories miraculously fail in describing certain cosmological effects. Among others, two particular problems produce a compelling mismatch between the SM and GR formulation and real world observations.

On the one hand, it is very well known that our universe only consists of around 5% baryonic matter, i.e. matter whose existence can be explained though SM effects. The remaining 95% consist of around 25% dark matter and roughly 70% dark energy which is a highly elusive form of matter and energy of unknown origin and whose existence is, so far, only supported by indirect experimental evidence. According to the logic of quantum mechanics both dark matter and dark energy must have a microscopic origin based on the existence and interaction of certain associated particles. But neither does the SM contain a good candidate for a dark matter particle nor does it contain any kind of explanation for dark energy. In fact, assuming that dark energy has its root in the presence of a cosmological constant, SM computations arrive at a mismatch of around 120 orders of magnitude between observation and prediction [6]. GR also fails at describing both dark matter and dark energy. This may be exemplified by the wrong GR prediction for the movement of stars in peripheral regions of galaxies if only the visible matter is taken into account [7].

On the other hand, no notion of gravity of any kind exists in the SM which is a highly undesired property from the point of view of a grand unification of all four forces.

It is believed that a unified theory combining GR and the SM into a quantum theory of gravity will resolve the tension between their theoretical predictions and dark matter/energy observations.

One possible approach to quantum gravity is an introduction of the graviton $h_{\mu\nu}$ in GR as the gravitational force mediator particle. This is done through a linear expansion of the GR metric tensor, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$ with κ being the coupling constant [8]. This approach is called graviton expansion and it defines a perfectly well defined quantum field theory of gravity which is non-renormalizable but fully capable of producing relevant gravitational corrections to physical observables if treated as an effective field theory [9]. Such a theory is called quantum GR (QGR). Scattering amplitudes can be computed as usual via the sum of all Feynman graphs with the complication that QGR contains an infinite amount of Feynman rules which are themselves gigantic expressions, e.g. the three graviton vertex contains 171 terms [10].

More theories of quantum gravity exist. The most prominent examples are loop quantum gravity, N = 8 supergravity (SUGRA) and superstring theory, which are all based on certain modifications of the SM or GR. Even though a large amount of research has been performed in those theories and they are of high relevance, we will refrain from diving any deeper into them and instead focus on quantum gravity theories which are as close as possible to QGR.

In 2010 a major breakthrough occurred regarding the calculation of amplitudes in QGR augmented by two particles, a real scalar field called dilaton and an antisymmetric two form called axion, i.e. $N = 0$ SUGRA. Z. Bern, H. Johansson and J. J. Carrasco realized that scattering amplitudes in such a theory can be constructed from pure YM amplitudes by a simple replacement rule [11–13]. Given a trivalent decomposition of a pure YM amplitude, achieved by decomposing every four gluon vertex into s, t, u channels, at any loop level L one can obtain the full $N = 0$ SUGRA amplitude by replacing color structures with kinematical ones, i.e.

$$\mathcal{A}^{(L)}_{\text{Gluons}} \propto \sum_{j\in\text{trivalent}} \prod_{i=1}^{L} \int d^D l_i \frac{c_j\, n_j}{D_j}$$

$$\mathcal{M}^{(L)}_{N=0\ ,\text{supergravity}} \propto \sum_{j\in\text{trivalent}} \prod_{i=1}^{L} \int d^D l_i \frac{n_j\, n_j}{D_j}\ ,$$

where c_j are color factors, n_j kinematic numerators and D_j propagator structures. This is true as long as the kinematic numerators are brought into a form that satisfies the same Jacobi relations as the color factors

$$c_i + c_j + c_k = 0 \iff n_i + n_j + n_k = 0\ .$$

This construction is called double copy and the kinematic numerators are said to be color dual. We will explore this relation more explicitly in the upcoming chapter.

At tree level, where pure YM amplitudes map directly to QGR amplitudes, the double copy has been proven [13] but it remains conjectural at loop level. Still, it exists a compelling amount of evidence supporting the validity of the double copy at highly non-trivial loop levels and between many more theories with and without supersymmetry [14–22, 22–27]. The current benchmark is its realization in $N = 8$ supergravity for the four point five-loop graviton scattering from gluon scattering in $N = 4$ super YM (SYM) [28]. Even though the technical aspect of the double copy is quite well under control it is not known what fundamental mechanisms are at work in this magical duality. A possible answer is to conclude an underlying Lie algebra from the Jacobi relations satisfying color dual numerators. This idea is called kinematic algebra and finding it might reveal hidden symmetries. So far the kinematic algebra has already been found in certain sub sectors of YM theory, i.e. in self-dual YM [29] and the non-linear sigma model [30].

It is a surprising observation that even though the double copy prescription is designed for scattering amplitudes, its core principles and results seem to also hold true in classical physical systems. In particular, it has been shown recently that the double copy also applies to classical perturbative solutions of YM and gravity equations of motion via a simple color-to-kinematics replacement rule [31–36] where even the notion of color dual numerators appears at some point [37]. This hints at a classical manifestation of the kinematic algebra. Therefore it is a particularly interesting connection due to a potentially new approach to the computations of gravitational radiation emitted by black hole binaries, i.e. gravitational waves.

In this thesis we focus on the double copy between classical effective actions of a system of two massive nonspinning classical worldlines. They are color charged and interact through pure YM on the gauge theory side, while they are dilaton charged and interact via dilaton gravity, i.e. $N = 0$ SUGRA without an axion, on the gravity side. Our goal is to extend the work of W. Goldberger and A. Ridgway on classical bound systems [33] by one further order in perturbation theory, while also to check the validity, and possibly to improve upon their proposed color-to-kinematics replacement rule. The key result shall be a proof of concept, pushing the boundaries of the double copy into more extreme settings, since effective actions are manifestly gauge variant objects in contrast to scattering amplitudes.

Aside from the double copy, another type of connections has emerged between color ordered tree level amplitudes in Einstein-YM (EYM), the graviton expanded version of diffeomorphism invariant pure YM theory, and pure YM theory. Such relations have been recently conjectured by S. Stieberger and T. Taylor [38, 39]. Their first connection concerns the finite contribution of the adjacent collinear gluon limit in pure YM amplitudes, i.e.

$$\frac{\kappa\, x}{\mathrm{g}^2} \sum_{\rho \in S_{n-3}} (-1)^{m_\rho}\, s_{\rho(n-2)p}\, \mathrm{A}_n^{\mathrm{YM}}(1, \rho(2), ..., \rho(n-2), n-1, n) \;=\; \mathrm{A}_{n-1}^{\mathrm{EYM}}(1, ..., n-2; p)\ . \tag{1.1}$$

Here A_n denotes color ordered amplitudes, p_n and p_{n-1} are the collinear gluons with limiting momentum p, $m_\rho = \{0, 1\}$ for even and odd numbers of permutations ρ respectively, x is the momentum fraction, g and κ are the couplings of gluons and gravitons respectively and $s_{ij} = 2\, p_i\, p_j$ are Mandelstam invariants. The right hand side, of the above equation, is an EYM amplitude where the collinear pair of gluons has been replaced by a graviton as indicated by the semicolon. The second connection is very similar to the first one with the modification that no collinear limit is needed, i.e.

$$\sum_{a=2}^{n} \varepsilon_p \cdot X_a\, \mathrm{A}^{\mathrm{YM}}(1, 2, ..., a, p, a+1, ..., n) = \frac{\kappa}{2\mathrm{g}^2}\, \mathrm{A}^{\mathrm{EYM}}(1, ..., n; p)\ , \tag{1.2}$$

where $X_a = \sum_{i=2}^{a} p_a$ is the region momentum and ε_p is the polarization vector of gluon p which is turned into a graviton on the rhs of this equation. We will discuss both formulae in the upcoming chapter in more detail. The main difference to the double copy prescription is that (1.1) (1.2) are relations between (collinear) gluons and single gravitons which do not seem to rely on any color-to-kinematics replacement rule. Furthermore, no general loop level conjecture exists for neither of the above formulae, although a conjecture at the level of loop level integrands has been formulated recently [40, 41].

The second relation (1.2) by S. Stieberger and T. Taylor has been proven and generalized to an arbitrary number of external gravitons [43–47] using a novel analytic representation of bosonic tree level amplitudes in arbitrary dimensions across a wide range of theories, the so-called Cachazo-He-Yuan (CHY) formalism [48–50] which we introduce in the upcoming chapter. This series of works has also sparked a general interest in EYM amplitudes at loop level [42, 51]. Relation (1.1) has neither been proven nor generalized to a generic number of gravitons. We will perform an analysis of both formulae.

We study the adjacent collinear gluon limit of color ordered pure YM amplitudes in the CHY formalism in order to compute the sub-leading adjacent collinear gluon behavior. Its generic analytic form is unknown but of high relevance for phenomenological high energy scattering predictions due to the role of collinear limits in the cancellation of infrared divergences in physical observables according to the Kinoshita-Lee-Nauenberg theorem [52, 53]. This will allow us to answer important questions regarding the universality and possible factorization properties of the sub-leading collinear gluon limit, both being unknown properties. Nevertheless, our main goal will be a generic proof of the first Stieberger-Taylor relation (1.1).

By analyzing the second Stieberger-Taylor relation (1.2), we prove its universality in the presence of massive quarks and scalars. This would be of interest in the context of computing loop level EYM amplitudes where unitarity methods involve computations of scattering amplitudes with fundamental matter, gluons and gravitons. Stieberger-Taylor relations would dramatically simplify computations in such scenarios. Furthermore, we aim at closing the gap between the double copy logic of a color-to-kinematics replacement rule and Stieberger-Taylor relations by trying to identify a new replacement rule.

This thesis is organized as follows. In the second chapter we introduce all relevant concepts and techniques for the following chapters. In the third chapter we dive into the computations of the sub-leading adjacent collinear gluon limit in the CHY formalism. This is followed by a proof of Stieberger-Taylor relations in this setup where we will encounter very intriguing and unintuitive characteristics. In the fourth chapter we prove universality of the second Stieberger-Taylor relation via a Feynman diagrammatic analysis which will yield a novel color-to-kinematics replacement rule. In the fifth chapter we compute the classical effective action of two massive worldline in YM and dilaton gravity. We will identify new color-to-kinematics replacement rules which allow us to observe a novel form of a classical double copy. Finally, we summarize and conclude in the sixth chapter.

Chapter 2

Aspects of tree level scattering amplitudes

In this chapter we will introduce the main technical and physical concepts, regarding scattering amplitudes in gauge and gravity theories, as a necessary basis for the upcoming chapters. The concepts in question range from very old and well known textbook techniques [54–57] to very modern techniques widely used in recent mathematical and physical research questions [58–60].

2.1 Tree level gluon scattering in Yang-Mills theory

Yang-Mills (YM) is a non-abelian gauge theory of a massless, colored, self interacting vector boson $A_\mu = A^a_\mu \, \mathrm{T}^a$ carrying two physical degrees of freedom, either $h = +1$ or $h = -1$ helicity. We use the standard textbook definition of the YM Lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\,\eta^{\mu\rho}\,\eta^{\nu\sigma}\,F^a_{\mu\nu}\,F^a_{\rho\sigma}\ , \tag{2.1}$$

where the gauge group is taken to be SU(N) with generators T^a and structure constants f^{abc} normalized as

$$\left[T^a, T^b\right] = i\,f^{abc}\,T^c \qquad , \qquad \mathrm{Tr}(T^a\,T^b) = \frac{1}{2}\,\delta^{ab}\ . \tag{2.2}$$

Let g be the YM coupling constant. Then we have

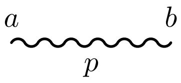
$$F^a_{\mu\nu} = \partial_\mu\,A^a_\nu - \partial_\nu\,A^a_\mu - i\,g\,f^{abc}\,A^b_\mu\,A^c_\nu\ . \tag{2.3}$$

We will refer to the gauge boson A^a_μ as the gluon (for $\mathrm{N} = 3$ this is the standard model gluon). Gluons are in the adjoint representation, their color indices run from $a = 1, ..., \mathrm{N}^2 - 1$.

We choose Feynman gauge with the gauge fixing Lagrangian

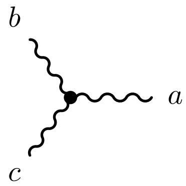
$$\mathcal{L}_{\mathrm{GF,YM}} = -\frac{1}{2}\,\left(\partial^\mu\,A^a_\mu\right)^2\ . \tag{2.4}$$

We will denote gluons as wiggly lines throughout this thesis. The propagator reads

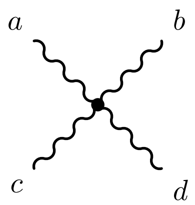


$$= \frac{-i\,\eta_{\mu_a\mu_b}\,\delta^{ab}}{p^2}\ , \tag{2.5}$$

and the vertices are given by (all momenta are ingoing)



$$= g\,f^{abc}\left[\eta^{\mu_a\mu_b}(p_a-p_b)^{\mu_c}+\eta^{\mu_b\mu_c}(p_b-p_c)^{\mu_a}+\eta^{\mu_c\mu_a}(p_c-p_a)^{\mu_b}\right]\ , \tag{2.6}$$



$$\begin{aligned} = -i\,g^2\Big[& f^{abe}\,f^{cde}\,(\eta^{\mu_a\mu_c}\eta^{\mu_b\mu_d}-\eta^{\mu_a\mu_d}\eta^{\mu_b\mu_c}) \\ & + f^{ace}\,f^{bde}\,(\eta^{\mu_a\mu_b}\eta^{\mu_c\mu_d}-\eta^{\mu_a\mu_d}\eta^{\mu_b\mu_c}) \\ & + f^{ade}\,f^{bce}\,(\eta^{\mu_a\mu_b}\eta^{\mu_c\mu_d}-\eta^{\mu_a\mu_c}\eta^{\mu_b\mu_d})\Big]\ . \end{aligned} \tag{2.7}$$

Every external gluon leg, in a given Feynman diagram, is contracted with a polarization vector $\varepsilon^{h_i}_\mu(p_i,r_i)=\varepsilon^{h_i}_{\mu,i}$. They are functions of their uniquely associated gluon momentum p_i, with helicity h_i, and reference momentum r_i. They satisfy the on-shell conditions

$$\varepsilon^{h_i}_i\cdot p_i=\varepsilon^{h_i}_i\cdot r_i=(\varepsilon^{h_i}_i)^2=0\ \ ,\ \ \varepsilon^+_i\cdot\varepsilon^-_i=-1. \tag{2.8}$$

We have ignored Feynman rules involving ghosts since we focus on tree level computations.

We denote n-point scattering amplitudes in Yang-Mills theory by $\mathcal{A}_n(\{p_i,\varepsilon_i\})$ which, at tree level, are rational functions of the external momenta and polarizations (we keep the dependence mostly implicit throughout this thesis and just write $\mathcal{A}_n$). Schematically, $\mathcal{A}_n$ takes the form

$$\mathcal{A}_n = \sum\{\text{all connected Feynman graphs}\} = \sum_{i\in\text{graphs}}\frac{\mathrm{C}_i\,\mathrm{N}_i}{\mathrm{D}_i}\ , \tag{2.9}$$

where we denote the color structure, of each single graph i, by C_i, the propagator structure by D_i and the kinematic numerator by N_i. Gauge invariance is encoded through invariance of the amplitude under a shift in any polarization $\varepsilon_i\to\varepsilon_i+p_i$. Note that each single term in the sum is, a priori, not gauge invariant.

There is, however, a way to decompose $\mathcal{A}^{\mathrm{YM}}_n$ into a linear combination of color structures times gauge invariant objects called partial amplitudes (also color ordered amplitude)[58].

This procedure is called color-decomposition and it is based on the successive use of two identities. On the one hand we use

$$\frac{i}{2} f^{abc} = \mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c) \ , \tag{2.10}$$

which follows directly from (2.2). On the other hand we use the SU(N) Fierz identity

$$(T^a)^i_j \, (T^a)^k_l = \frac{1}{2} \left(\delta^i_l \, \delta^k_j - \frac{1}{\mathrm{N}} \, \delta^i_j \, \delta^k_l \right) \ , \tag{2.11}$$

which follows directly from the fact that the generators T^a together with $1_{\mathrm{N}\times\mathrm{N}}$ form a complete basis of hermitian matrices of U(N).

Color decomposition starts with the use of (2.10) inside of (2.9). This will decompose a generic color structure C_i into a sum of products of traces. Internally transported color, e.g. with color index a, manifests itself in products

$$\mathrm{Tr}(...T^a...) \, \mathrm{Tr}(...T^a...) \ . \tag{2.12}$$

We use the Fierz identity to eliminate the internal color at the cost of combining the product of two traces into one trace. Repeating this procedure will eliminate all internal colors and yield traces that exclusively contain generators of the external colors. This defines the color basis

$$\mathcal{A}_n = \sum_{\sigma \in \mathcal{S}_{n-1}} \mathrm{Tr}(T^{a_1} \, T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}}) \, \mathrm{A}_n(1, \sigma(2), \sigma(3), ..., \sigma(n)) \ , \tag{2.13}$$

where the first index is fixed by using trace cyclicity. The objects A_n are partial amplitudes and they are the starting point of many remarkable advances in high-energy physics [60–63]. We will also denote them by $\mathrm{A}_n^{\mathrm{YM}}$ if required. Note that the second term in the Fierz identity (2.11) does not contribute in the color decomposition at tree level. It is possible to observe the cancellation explicitly but it can also be intuitively understood since gluons do not couple to photons which are associated to this term. We stress that this is only true at tree level.

Partial amplitudes satisfy additional properties

1. **Cyclicity:**

$$\mathrm{A}_n(1, 2, 3, ..., n) = \mathrm{A}_n(2, 3, ..., n, 1) = \mathrm{A}_n(3, ..., n, 1, 2) = ... \ . \tag{2.14}$$

2. **Reflectivity:**

$$\mathrm{A}_n(1, 2, 3, ..., n-1, n) = (-1)^n \, \mathrm{A}_n(n, n-1, ..., 3, 2, 1) \tag{2.15}$$

3. **Kleiss-Kuijf (KK) relations:** The first non-trivial relations were proposed in [64] and proven in [65]. They read

$$\mathrm{A}_n(1, B, n, C) = (-1)^{|C|} \sum_{\sqcup} \mathrm{A}_n(1, B \sqcup C^T, n) \ , \tag{2.16}$$

for any set of labels $B = \{b_1, ..., b_N\}$, $C = \{c_1, ..., c_M\}$ where $C^T = \{c_M, ..., c_1\}$ is the reverse ordering of C and the shuffle product $B \sqcup C^T$ is defined as some ordering of

$B \cup C^T$ which keeps the individual orderings of B and C^T unchanged. The sum runs over all shuffles, e.g.

$$\sum_{\sqcup\!\sqcup} \{2,3\} \sqcup\!\sqcup \{5,4\}$$
$$= \{2,3,5,4\} + \{2,5,3,4\} + \{5,2,3,4\} + \{5,2,4,3\} + \{5,4,2,3\} + \{2,5,4,3\} \; . \tag{2.17}$$

Finally, $|C|$ denotes the cardinality of set C. A special case of the KK-relations is traditionally treated distinct and occurs if one of the sets has unit length, e.g. $C^T = \{c\}$. Then we arrive at

$$\mathrm{A}_n(1,B,n,c) + \mathrm{A}_n(1,c,B,n) + \sum_{i=1}^{|B|} \mathrm{A}_n(1,...,b_i,c,b_{i+1},...,n) = 0 \; , \tag{2.18}$$

referred to as U(1), or photon, decoupling.

4. **Bern-Carrasco-Johansson (BCJ) relations:** Another set of highly non-trivial, non-intuitive relations were discovered in [11] and proven in [66]. For a canonical ordering of the arguments, i.e. $\{1,2,...,n\}$, they read

$$0 = \sum_{i=2}^{n} p_k \cdot \left(\sum_{m=2}^{i} p_m \right) \mathrm{A}_{n+1}(1,2,...,i,k,i+1,...,n) \; , \tag{2.19}$$

where we identify the indices $n+1=1$. Often, the notion of a region momentum is introduced for more compact expressions

$$X_k = \sum_{i=2}^{k} p_i \; , \tag{2.20}$$

i.e. the sum of all momenta in front of p_k, except p_1[1], inside of the argument of the scattering amplitude. These identities generalize trivially to any set of labels

$$0 \;=\; \sum_{\sqcup\!\sqcup} \; p_k \cdot X_k \, \mathrm{A}_{n+1}(1, \{B\} \sqcup\!\sqcup \{k\}) \quad , \quad \forall k \in \{1,...,n+1\} \; , \tag{2.21}$$

where B is some permutation of $\{2,...,n\}$ not including k. Here, the notion of a region momentum is truly advantageous.

It is worthwhile to discuss a subtle difference between BCJ relations and KK relations. KK relations are group theoretic identities, i.e. identities which only involve permutations of the arguments inside an amplitude. BCJ relations, however, do also involve on-shell kinematic prefactors for every amplitude in the linear combination. Both KK and BCJ relations can be used in the color decomposition (2.13) such that the number of terms is reduced from $(n-1)!$ down to, at most, $(n-3)!$ terms.

[1]We can relabel the indices such that p_1 is also included. Then the region momentum is the sum of all momenta in the amplitudes argument before p_k. We can also use momentum conservation and change the region momentum to include all momenta after p_k.

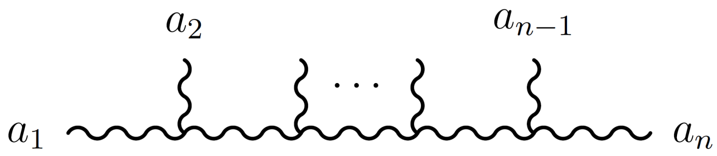


Figure 2.1: A half-ladder diagram associated with the color structure of the DDM color decompositon.

Similar to [67], we will call a color decomposition proper if all group theoretic relations have been used. In the case of YM amplitudes (2.13) is not a proper decomposition. However, we can use KK relations (2.16) to fix a second label at some position, thus obtaining a proper decomposition

$$\mathcal{A}_n = \sum_{\sigma \in \mathcal{S}_{n-2}} \mathrm{Tr}(\mathrm{T}^{a_1}\, \mathrm{T}^{a_{\sigma(2)}}\, ...,\, \mathrm{T}^{a_{\sigma(n-1)}}\, \mathrm{T}^{a_{(n)}})\, \mathrm{A}_n(1, \sigma(2), ..., \sigma(n-1), n)\ , \tag{2.22}$$

which we will call KK basis. Another version of the same proper decomposition is the Del-Duca-Dixon-Maltoni (DDM) basis [65]. The idea is to keep the color structures in (2.9) expressed in terms of structure constants f^{abc} and reduce their amount by using Lie algebra Jacobi relations

$$f^{ade}\, f^{bcd} + f^{bde}\, f^{cad} + f^{cde}\, f^{abd} = 0\ , \tag{2.23}$$

until the color structures correspond to half-ladder diagrams, see figure 2.1, e.g.

$$\mathrm{C}_{\mathrm{DDM}}(1, 2, ..., n-1, n)\ =\ f^{a_1 a_2 b_1}\, f^{b_1 a_3 b_2}\, f^{b_2 a_4 b_3}\, \cdots f^{b_{n-2} a_{n-1} a_n}\ , \tag{2.24}$$

where the labels a_i represent the colors of the external particles. The DDM basis then reads

$$\mathcal{A}_n = \sum_{\sigma \in S_{n-2}} C_{\mathrm{DDM}}(1, \sigma(2), ..., \sigma(n-1), n)\, \mathrm{A}_n(1, \sigma(2), ..., \sigma(n-1), n)\ . \tag{2.25}$$

The coefficients A_n are the same partial amplitudes as in (2.13). Note that, with the help of proper decompositions, we can interpret BCJ relations (2.19) as relations among partial amplitudes in a proper decomposition. Even though interesting in itself, we will not discuss improper decompositions, i.e. decompositions where three indices are fixed at some positions with the help of BCJ relations.

We end this chapter by discussing an important feature of gauge invariance in YM theory. On the level of scattering amplitudes gauge invariance is realized as a shift of any polarization vector by its momentum which leaves the amplitude unchanged. The same holds true for partial amplitudes, i.e.

$$\begin{aligned} &\varepsilon_i^\mu \to \varepsilon_i^\mu + p_i^\mu \ =: \tilde{\varepsilon}_i^\mu \\ &\Rightarrow \mathrm{A}_n(1, ..., i, ..., n) = \varepsilon_i^\mu\, \mathrm{A}_{n,\mu}(1, ..., i, ..., n) = \tilde{\varepsilon}_i^\mu\, \mathrm{A}_{n,\mu}(1, ..., i, ..., n)\ , \end{aligned} \tag{2.26}$$

which implies

$$p_i^\mu\, \mathrm{A}_{n,\mu}(1, ..., i, ..., n) = 0\ . \tag{2.27}$$

The vector $\mathrm{A}_{n,\mu}$ denotes the scattering amplitude where leg i is not contracted with a polarization. Eq. (2.27) is traditionally referred to as a Ward-identity[2]. The crucial point in (2.27) regards its interplay with on-shell properties. All legs except the shifted leg have to be on shell [56],[57] [3]

2.2 Scattering Amplitudes in (Scalar-) Quantum Chromo Dynamics

YM theory by itself is of utmost interest in modern scattering amplitudes research. Nevertheless, we are also interested in the coupling of gluons to matter. In particular, we want to establish the interaction between gluons, complex scalars and spin $\frac{1}{2}$ fermions. The goal of this section is to introduce a proper color decomposition for such scattering scenarios.

The theory we consider is given by the Lagrangian [56, 57]

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{sQCD}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{GF,YM}} \ , \tag{2.28}$$

with the new terms

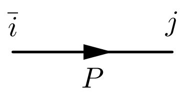
$$\mathcal{L}_{\text{QCD}} = \bar{\psi} \left(i \not{D} - m_{\psi} \right) \psi \tag{2.29}$$

$$\mathcal{L}_{\text{sQCD}} = \eta^{\mu\nu} \left(D_{\mu} \phi \right)^{\dagger} \left(D_{\nu} \phi \right) - m_{\phi}^{2} \phi^{\dagger} \phi \ , \tag{2.30}$$

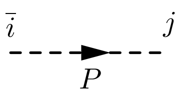
where we used the Feynman-slash notation $\not{V} = \gamma_{\mu} V^{\mu}$. The first Lagrangian $\mathcal{L}_{QCD}$ describes Quantum-Chromo-Dynamics (QCD), i.e. the interaction between gluons and massive fundamentally charged quarks. The second Lagrangian $\mathcal{L}_{sQCD}$ describes Scalar-Quantum-Chromo-Dynamics (sQCD), i.e. the interaction of gluons and fundamentally charged massive complex scalars. The gauge group is SU(N) and the coupling is encoded in the covariant derivative

$$D^{\mu}_{ij} = \partial_{\mu} \delta_{ij} - i \, g \, \mathrm{A}^{\mu}_{a} \, (T^{a})_{ij} \ . \tag{2.31}$$

Note that in the above theory, in contrast to the Standard model, the quarks and scalars are only charged under one gauge group, i.e. it only mimics the mathematical structure of the strong force. In addition to the YM Feynman rules given in section 2.1, we have the new propagators



$$= \frac{i \, \delta^{j \bar{i}}}{\not{P} - m_{\psi}} \ , \tag{2.32}$$

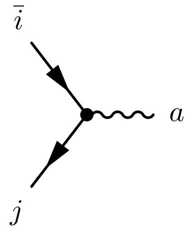


$$= \frac{i \, \delta^{j \bar{i}}}{P^{2} - m_{\phi}^{2}} \ , \tag{2.33}$$

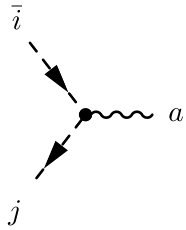
[2] Technically speaking it is a consequence of Slavnov-Taylor identities [68],[69].

[3] This is in contrast to QED amplitudes where only the Fermion momenta have to be on-shell.

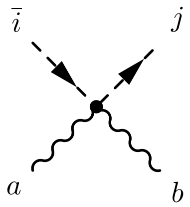
and the vertices (all momenta are ingoing)



$$= i\, g\, \gamma^{\mu_a} (T^a)_{j\bar{i}} \ , \tag{2.34}$$



$$= i\, g (p_i - p_j)^{\mu_a} \, (T^a)_{j\bar{i}} \ , \tag{2.35}$$



$$= i\, g^2 \eta^{\mu_a \mu_b} \{T^a, T^b\}_{j\bar{i}} \ , \tag{2.36}$$

where we denote quarks as solid lines, scalars as dashed lines and gluons as wiggly lines. External gluons are contracted with their polarization vectors, scalars are not contracted with anything and an external quark/antiquark line, which must start and end externally in a tree level diagram, is sandwiched in between appropriate spinors $\bar{u}(p), \bar{v}(p), u(p), v(p)$ depending the type and directions of the quark/antiquark.

The question of a tree level proper color decomposition in (2.28) has only recently been addressed in [26, 70] and proven [67]. The construction is much more involved compared to the pure YM case due to new color structures given by products of the SU(N) generators with open quark/antiquark (or scalar/antiscalar) flavor indices, i.e.

$$(T^{a_1} \, T^{a_2} \cdots T^{a_k})_{\bar{i}j} \ . \tag{2.37}$$

We call the resulting proper decomposition Melia-Johansson-Ochirov (MJO) basis and we will discuss it for QCD scattering amplitudes while keeping the straightforward sQCD generalization in mind.

The formula for n external particles with k quark/antiquark pairs and $n-2k$ gluons reads

$$\mathcal{A}_{n,k} = \sum_{\sigma \in \text{MJO basis}}^{\chi(n,k)} C(\underline{1}, \bar{2}, \sigma) \, \mathrm{A}^{\mathrm{QCD}}(\underline{1}, \bar{2}, \sigma) \ , \tag{2.38}$$

where $\chi(n,k) = \frac{(n-2)!}{k!}$ is the dimension of the basis. Quarks and antiquarks are marked with under-scores or over-scores respectively. Without loss of generality we take the flavors of all k quark lines to be distinct. We observe that two legs have been fixed in the partial amplitudes $\mathrm{A}(\underline{1},\bar{2},\sigma)$. This introduces a preferred quark/antiquark line in all diagrams. We shall call this line base quark line. The MJO decomposition is almost equivalent to a YM amplitude $\mathcal{A}_n$ in the KK-/DDM-basis (2.22)(2.25) except that the fixed quark labels must stay adjacent. The set of partial amplitudes in the MJO-basis defines the permutation σ

$$\left\{\mathrm{A}^{\mathrm{QCD}}(\underline{1},\bar{2},\sigma)\middle|\sigma\in\mathrm{Dyck}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\right\}.$$

More explicitly, the last $n-2$ arguments of the partial amplitudes have to form a valid Dyck-word which encodes the distribution of all quark/antiquark lines in the diagrams, except for the base quark line.

One way to define valid Dyck-words in this context is to assign a bracket to each quark line, i.e. a bra "$\{i|$" to a quark with flavor a_i and a ket "$|i+1\}$" to the corresponding antiquark with the same flavor. We will later show how to associate bracket expression to color structures by using associations of the form

$$\{i|(T^{a_i})|i+1\}=(T^{a_i})_{k_i\bar{k}_{i+1}}\ . \tag{2.39}$$

Note that in our notation of the amplitude, we label the quarks by their momentum but implicitly group them by their flavor. More explicitly, a quark with some unique flavor carries an odd value label and the corresponding anti-quark carries an even value label obtained from the odd value by an increase of one. For example, the base quark line $\underline{1}\leftarrow\bar{2}$ has flavor a_1 and is by convention complex conjugated. Another one could be $\underline{5}\rightarrow\bar{6}$ with flavor a_5 and so forth. A valid Dyck word is then a sequence of bras and kets such that at any point in the sequence the number of bras is bigger or equal to the number of kets and every bra is closed by the uniquely associated ket. For example take an amplitude with $k=3$,$n=6$, i.e. three quark lines with flavors a_1,a_3 and a_5. The permutations σ in the amplitudes arguments are then formed from bras and kets $\{3|$,$\{5|$, $|4\}$,$|6\}$ with the valid Dyck words

$$\{3|4\}\{5|6\}\ ,\ \{3|\{5|6\}4\}\ ,\ \{5|6\}\{3|4\}\ ,\ \{5|\{3|4\}6\}\ , \tag{2.40}$$

an the corresponding partial amplitudes are then

$$\mathrm{A}(\underline{1},\bar{2},\underline{3},\bar{4},\underline{5},\bar{6})\ ,\ \mathrm{A}(\underline{1},\bar{2},\underline{3},\underline{5},\bar{6},\bar{4})\ ,\ \mathrm{A}(\underline{1},\bar{2},\underline{5},\bar{6},\underline{3},\bar{4})\ ,\ \mathrm{A}(\underline{1},\bar{2},\underline{5},\underline{3},\bar{4},\bar{6})\ . \tag{2.41}$$

An example for an invalid Dyck word would be $\{3|\{5|4\}6\}$ since the bra with flavor a_5 cannot be closed with a ket of flavor a_3.

The other part in σ are the gluon insertions. Given a valid Dyck word, gluons are inserted at any possible position except between the base quark line $\underline{1}\leftarrow\bar{2}$. Thus a generic partial amplitude may look like

$$\mathrm{A}(\underline{1},\bar{2},\underline{3},7,8,\underline{5},9,\bar{6},\bar{4})\ ,$$

with gluons 7,8 and 9. The corresponding valid Dyck word would be

$$\sigma=\{3|78\{5|9|6\}|4\}\ .$$

We now turn to the MJO color structure in (2.38). It is given by [26]

$$C(\underline{1},\bar{2},\sigma)=(-)^{k-1}\,\{2|\sigma|1\}\ \left|\begin{array}{l}q\rightarrow\{q|\,T^b\otimes\Xi^b_{l-1}\\ \bar{q}\rightarrow|q\}\\ g\rightarrow\Xi^{a_g}_l\end{array}\right.\ . \tag{2.42}$$

The Dyck word σ is inserted in terms of brackets and gluons labels turning the color structure into a valid Dyck word which is then read from left to right and every time one encounter a bra, which is not $\{2|$, or a gluon label the given replacement rule is used. This is straightforward once we define

$$\Xi^a_l = \sum_{s=1}^{l} \underbrace{1 \otimes \ldots \otimes 1 \otimes \overbrace{T^a \otimes 1 \otimes \ldots 1 \otimes \bar{1}}^{s}}_{l}, \tag{2.43}$$

and also define l as a level of nestedness in the Dyck word which is the number of bras minus the number of kets to the left of the position in the Dyck word. One also defines $\overline{T}^a_{\bar{j}i} := -T^a_{i\bar{j}}$ in analogy to $f^{abc} = -f^{cba}$. Note that the Ξ^a_l form a representation of the gauge group Lie algebra

$$[\Xi^a_l, \Xi^b_l] = \tilde{f}^{abc}\Xi^c_l \quad , \quad \tilde{f}^{abc} = i\, f^{abc} \tag{2.44}$$

with the introduction of a normalized structure constant. We can also see explicitly that the outermost bracket is $\{2|1\}$, the complex conjugate of $\{1|2\}$ and mere convention. For the same reason we see that the last tensor product in Ξ^a_l is barred.

Let us explain the finer technical details of (2.42) through examples. First consider

$$\begin{aligned} C(\underline{1}, \bar{2}, \underline{3}, \bar{4}, \underline{5}, \bar{6}) &= \{2|\{3|T^{a_3} \otimes \Xi^{a_3}_1|4\}\{5|T^{a_5} \otimes \Xi^{a_5}_1|6\}|1\} \\ &= \{2|\{3|T^{a_3} \otimes \bar{T}^{a_3}_1|4\}\{5|T^{a_5} \otimes \bar{T}^{a_5}_1|6\}|1\} \\ &= \{2|\bar{T}^{a_3}\bar{T}^{a_5}|1\}\{3|T^{a_3}|4\}\{5|T^{a_5}|6\} = (T^{a_5}T^{a_3})_{i_1\bar{i}_2}\, T^{a_3}_{i_3\bar{i}_4}\, T^{a_5}_{i_5\bar{i}_6}\ . \end{aligned} \tag{2.45}$$

Here we learn that the tensor product moves Lie algebra generators from the most inner layer of brackets into outer layers of brackets. Note that due to the structure of Ξ_l, the last tensor product has to be barred since it always ends up in the outermost layer $\{2|1\}$. Thus $\{2|1\}$ will only contain complex conjugates of the Lie generators and in the last step this bracket is complex conjugated. Finally, one uses the association (2.39).

The second example includes a gluon

$$\begin{aligned} C(\underline{1}, \bar{2}, \underline{3}, 5, \bar{4}) &= -\{2|\{3|(T^{a_3} \otimes \Xi^{a_3}_1)\Xi^{a_5}_2|4\}|1\} \\ &= -\{2|\{3|(T^{a_3} \otimes \bar{T}^{a_3})(1 \otimes \bar{T}^{a_5})|4\}|1\} - \{2|\{3|(T^{a_3} \otimes \bar{T}^{a_3})(T^{a_5} \otimes \bar{1})|4\}|1\} \\ &= -\{2|\bar{T}^{a_3}\, \bar{T}^{a_5}|1\}\{3|T^{a_3}|4\} - \{2|\bar{T}^{a_3}|1\}\{3|T^{a_3}\, T^{a_5}|4\} \\ &= -(T^{a_5}\, T^{a_3})_{i_1\bar{i}_2}\, (T^{a_3})_{i_3\bar{i}_4} + (T^{a_3})_{i_1\bar{i}_2}\, (T^{a_3}\, T^{a_5})_{i_3\bar{i}_4}\ . \end{aligned} \tag{2.46}$$

Now that we understand the MJO color decomposition we want to give its most remarkable, surprising and relevant result for the context of this thesis, namely that the partial amplitudes satisfy the exact same BCJ relations for every external gluon present in the scattering process [26], similar to partial YM amplitude, i.e.

$$0 = \sum_{\sqcup\!\sqcup} p \cdot X_p \mathrm{A}^{\mathrm{QCD}}(\underline{1}, \bar{2}, \{\sigma\} \sqcup\!\sqcup\, p) \tag{2.47}$$

where p is a gluon momentum. This formula was proven in [71, 72] and it is in perfect agreement with the previously introduced BCJ relations for YM amplitudes (2.19).

We end this section by stressing that the exact same decomposition is straightforwardly obtained for sQCD. The only adjustment that has to be done is to replace the notion of a base quark line by a base scalar line. Furthermore, note that at no point in this section have we explored how to actually compute the partial amplitudes in the MJO basis. This is due to the exclusive desire for a proper color decomposition which will be of critical importance in later chapters. Finally, we want to note that the partial amplitudes do satisfy Ward identities for every gluon in the scattering process similar to (2.27), i.e. the shifted leg does not have to be on-shell while all other particles must be on-shell.

2.3 Graviton Scattering Amplitudes in Einstein-Gravity

In this section we want to introduce the notion of a graviton and cover the basics of scattering amplitudes involving gravitons. We begin by writing down the Einstein-Hilbert Lagrangian[4]

$$\mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2}\sqrt{-g}\, g^{\mu\nu}\, R_{\mu\nu} \;, \tag{2.48}$$

where $R_{\mu\nu}$ is the usual Ricci tensor and κ is the gravitational coupling constant which is defined as the inverse of the Planck mass m_{Pl}, i.e.

$$\kappa = m_{Pl}^{-1} = \sqrt{32\,\pi\, G} \;, \tag{2.49}$$

G being the Newton constant. The tensor $g_{\mu\nu}$ is a metric of curved spacetime and $\sqrt{-g} = \sqrt{\det(g_{\mu\nu})}$. The Ricci tensor is obtained from the Riemann tensor by contraction of two indices

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \;, \tag{2.50}$$

where the Riemann tensor is defined via the Christoffel symbols

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta\, \Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu\, \Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\beta\sigma}\, \Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\sigma}\, \Gamma^{\sigma}{}_{\mu\beta} \;, \tag{2.51}$$

and finally the Christoffel symbols are given by

$$\Gamma^{\alpha}{}_{\mu\nu} = g^{\alpha\beta}\, \Gamma_{\beta\mu\nu} = \frac{1}{2}\, g^{\alpha\beta}\left(\partial_\nu\, g_{\beta\mu} + \partial_\mu\, g_{\beta\nu} - \partial_\beta\, g_{\mu\nu}\right) \;. \tag{2.52}$$

The Einstein-Hilbert Lagrangian defines the theory of general relativity (GR) which, with incredible accuracy, describes gravity in nature as has been recently verified again through measurements of gravitational waves by the LIGO and Virgo collaboration (see e.g. [3]).

We want to study (2.48) in a perturbative setting to extract the tree level scattering amplitudes. Even though there exists a high desire, it is very well known and understood that standard quantization procedures are impossible to implement in (2.48), which is sometimes referred to as the incompatibility of QFT and GR [75]. A circumvention of this problem is

[4]Technically speaking, we have to add a term $\mathcal{L}_{\text{GHY}}$ which denotes the Gibbons-Hawking-York boundary term [73, 74]. This guarantees vanishing surface terms in a mathematically rigorous way.

to imagine that the full dynamical metric $g_{\mu\nu}$ can be separated into a flat, static part and a dynamical part

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x) \;, \tag{2.53}$$

where $h_{\mu\nu}$ is known as the graviton and κ is the gravitational coupling from (2.48). This ansatz is known as graviton expansion. We can expand the Einstein-Hilbert Lagrangian in terms of gravitons obtaining a perfectly quantizable field theory of a self interacting, massless, symmetric spin 2 field in Minkowski background. We call the resulting theory quantum general relativity (QGR).

There are three subtle and potentially problematic features we need to discuss. First, and most obvious, QGR will be non-renormalizable from a power counting point of view due to the negative mass dimension of κ. This issue is very well known and usually referred to when talking about GR being non-renormalizable, rejecting it from being a quantum theory of gravity realized in nature. While problematic, one may argue that we are only interested in the effective field theory (EFT) sector where we are restricting the dynamics up to some energy scale well below κ^{-1} and thus avoid renormalization issues [9]. This is a reasonable approach due to the monstrous magnitude of $\kappa^{-1} = \mathcal{O}(10^{19} GeV)$. Aside from that, it is by itself interesting to study the divergent structure of QGR due to the general interest in a consistent quantum theory of gravity. For example, it is possible to show the, non-intuitive and surprising, result that QGR is one-loop renormalizable [76].

The second subtlety is that our perturbation parameter will necessarily be κ. This is fine on first glance due to its size $\kappa \propto 10^{-19}\text{GeV}^{-1}$ but very problematic once realized that its size is strongly dependent on the system of units used. A better perturbation parameter would be $\frac{\kappa}{\lambda_{\text{C}}}$ for some characteristic wavelength or size λ_{C} of the system in question. We keep this fact implicit and will view the expansion in κ as formal.

The last issue we want to mention is that in order to plug in (2.53) into (2.48) we also need to expand the inverse metric $g^{\mu\nu}$ and the square root of the determinant $\sqrt{-g}$ in terms of the graviton field. This is problematic because expanding an inverse and a square root always produces an infinite number of terms leading to infinitely many Feynman rules. Additionally, all Feynman rules are highly complex due to the sheer amount of terms [10]. This will provide an obstacle in future chapters.

Let us now establish the Lagrangian for QGR. We start with the expansion of the various building blocks in the Einstein-Hilbert Lagrangian, i.e.

$$\begin{aligned} \sqrt{-g} &= 1 + \frac{\kappa}{2}\, h + \frac{\kappa^2}{8}\left(h^2 - 2h_{\alpha\beta}\, h^{\alpha\beta}\right) + \mathcal{O}(\kappa^3) \;\;, \\ g^{\mu\nu} &= \eta^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2\, h^{\mu\alpha}\, h_{\alpha}^{\;\nu} + \mathcal{O}(\kappa^3) \;\;, \end{aligned} \tag{2.54}$$

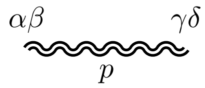
where we introduced the trace of the graviton as $h = h^{\mu}_{\mu}$ and we extract the inverse expansion of the metric tensor through the condition $g_{\mu\alpha}\, g^{\alpha\nu} = \delta^{\nu}_{\mu}$. The full Lagrangian is then straightforward to obtain and the leading order is given by

$$\mathcal{L}_{\text{EH}} = \frac{1}{2}\,\partial_\mu\, h_{\alpha\beta}\,\partial^\mu\, h^{\alpha\beta} - \frac{1}{2}\,\partial_\mu\, h\,\partial^\mu\, h + \partial_\mu\, h^{\mu\nu}\,\partial_\nu\, h - \partial_\mu\, h^{\mu\nu}\,\partial^\rho\, h_{\rho\nu} + \mathcal{O}(\kappa) \;, \tag{2.55}$$

where we omit higher order terms to keep the notation compact. We gauge fix diffeomorphism invariance using De-Donder gauge

$$\mathcal{L}_{\mathrm{GF}} = \left(\partial^{\nu} h_{\mu\nu} - \frac{1}{2} \partial_{\mu} h^{\alpha}_{\alpha} \right)^2 \tag{2.56}$$

and again ignore any presence of ghosts. The propagator is then given by



$$= \frac{i\, P_{\alpha\beta\gamma\delta}}{p^2} \tag{2.57}$$

where we defined

$$P_{\alpha\beta\gamma\delta} = \eta_{\alpha(\beta}\eta_{\gamma)\delta} - \frac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta} \quad , \quad \eta_{\alpha(\gamma}\eta_{\delta)\beta} = \frac{1}{2}\left(\eta_{\alpha\gamma}\,\eta_{\delta\beta} + \eta_{\alpha\delta}\,\eta_{\gamma\beta}\right) \ , \tag{2.58}$$

with d being the spacetime dimension.

Going further immediately confronts us with an obstacle. The three graviton vertex will have 171 terms [10] and the four graviton vertex around 1000. We will refrain from writing out the results explicitly, since we will be able to explore simplifications and practicalities in later chapters. At this point we want to complete the Feynman rules by giving the external leg rule. Every incoming external graviton is contacted with its polarization tensor $\varepsilon_i^{\mu\nu}(p,r,q)$ where the corresponding graviton carries momentum p and the polarization also depends on two arbitrary reference momenta q,r. Outgoing gravitons are contracted with the conjugate tensor. The graviton is a massless, traceless, symmetric spin 2 particle and thus carries two degrees of freedom given by its helicity states which are either $+2$ or -2. It is clearly possible to encode this structure into a representation of the graviton polarization tensor via a product of gluon polarization vectors, i.e.

$$\varepsilon_i^{\mu\nu}(p,r,q) = \varepsilon_i^{\mu}(p,q)\, \varepsilon_i^{\nu}(p,r) \ . \tag{2.59}$$

The polarization tensor satisfies on-shell conditions inherited from (2.8) and it is clearly traceless.

Scattering amplitudes can be computed straightforwardly but with an enormous effort due to the size of intermediate expressions. They will be of the schematic structure

$$\mathcal{M}_n(1,..,n) = \sum_{i \in \text{graphs}} \frac{\tilde{\mathrm{N}}_i}{\mathrm{D}_i} \ , \tag{2.60}$$

where $\tilde{\mathrm{N}}_i$ are the kinematic numerators obtained from Feynman rules. They have two powers of polarization vectors for every leg and a unit dimension twice the size of a YM numerator in (2.9), i.e. $[\tilde{\mathrm{N}}_i] = 2\,[\mathrm{N}_i]$. The D_i are again the propagator structures.

Since gravitons are not charged under any non-abelian group, no ordering preferences exist in the amplitude and thus no partial amplitudes have to be introduced.

Regarding Ward identities (2.27) the trivial generalization holds, i.e. for every polarization tensor $\varepsilon_i^{\mu}(p,q)\,\varepsilon_i^{\nu}(p,r)$ we can shift either polarization vector by its momentum without changing the amplitude. Again, the shifted leg does not have to be on-shell compared to all other particles in the scattering.

2.4 Scattering between Gravitons and Matter

In this section we want to introduce the coupling of gravitons to fermionic and bosonic matter. In particular, we want to couple gluons, massive quarks and massive fundamental scalars to gravitons. We will finish this section by modifying the previously introduced proper color decompositions to scattering scenarios including gravitons.

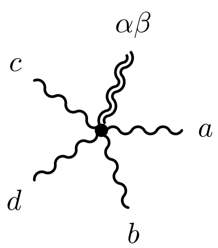
We start with the straightforward implementations of gravitons in bosonic QFTs, i.e. YM and sQCD. The first step is to take the Lagrangian of such a theory and promote its coordinate symmetries to diffeomorphism invariance through the standard replacement [75, 77]

$$\mathcal{L}(\eta, \partial) \to \sqrt{-g}\, \mathcal{L}(g, \nabla) + \mathcal{L}_{\text{EH}} \ , \tag{2.61}$$

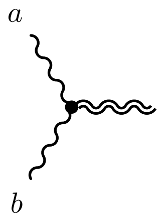
where we only focus on the Minkowski metric dependence of the theory which we then replace by a generic dynamical metric, multiply the Lagrangian by $\sqrt{-g}$ and replace any partial derivative by a covariant derivative which involves a spin connection. Finally, we add the Einstein-Hilbert action (2.48). Doing so for YM and sQCD yields their generalization into curved spacetime

$$\begin{aligned} \mathcal{L}_{\text{YM}} &= -\frac{1}{4}\, \eta^{\mu\rho}\, \eta^{\nu\sigma}\, \mathrm{F}^{a}_{\mu\nu}\, \mathrm{F}^{a}_{\rho\sigma} \\ &\to -\frac{\sqrt{-g}}{4}\, g^{\mu\rho}\, g^{\nu\sigma}\, \mathrm{F}^{a}_{\mu\nu}\, \mathrm{F}^{a}_{\rho\sigma} + \mathcal{L}_{\text{EH}} \ , \\ \mathcal{L}_{\text{sQCD}} &= \eta^{\mu\nu}\, (D_\mu\, \phi)^\dagger\, (D_\nu\, \phi) - m_\phi^2\, \phi^\dagger\, \phi \\ &\to \sqrt{-g}\, \Big(g^{\mu\nu}\, (D_\mu\, \phi)^\dagger\, (D_\nu\, \phi) - m_\phi^2\, \phi^\dagger\, \phi \Big) + \mathcal{L}_{\text{EH}} \ . \end{aligned} \tag{2.62}$$

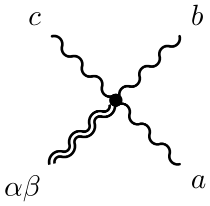
The second step is to plug in the graviton expansion (2.53), thus introducing gravitons. At this point the construction is finished. Gauge fixings are only done at this point. The resulting theories are called Einstein-Yang-Mills (EYM) and Einstein-scalar-quantum-chromodynamics (EsQCD). Both are power counting non-renormalizable and of high complexity due to the appearance of graviton self interactions and infinite many Feynman rules. Using Feynman gauge for gluons and De-Donder gauge for gravitons, we obtain vertices with up to one graviton carrying momentum p



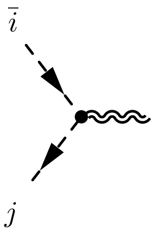
$$\begin{aligned} = (ig^2\kappa)\Big[& f^{abe} f^{cde} (P^{\alpha\beta\mu_a\mu_c}\eta^{\mu_b\mu_d} + P^{\alpha\beta\mu_b\mu_d}\eta^{\mu_a\mu_c} - P^{\alpha\beta\mu_b\mu_c}\eta^{\mu_a\mu_d} \\ & - P^{\alpha\beta\mu_a\mu_d}\eta^{\mu_b\mu_c}) + f^{ace} f^{bde} (P^{\alpha\beta\mu_a\mu_b}\eta^{\mu_c\mu_d} + P^{\alpha\beta\mu_c\mu_d}\eta^{\mu_a\mu_b} \\ & - P^{\alpha\beta\mu_c\mu_b}\eta^{\mu_a\mu_d} - P^{\alpha\beta\mu_a\mu_d}\eta^{\mu_b\mu_c}) + f^{ade} f^{bce} (P^{\alpha\beta\mu_a\mu_c}\eta^{\mu_b\mu_d} \\ & + P^{\alpha\beta\mu_d\mu_b}\eta^{\mu_a\mu_c} - P^{\alpha\beta\mu_d\mu_c}\eta^{\mu_a\mu_b} - P^{\alpha\beta\mu_a\mu_b}\eta^{\mu_d\mu_c}) \Big] \ , \end{aligned} \tag{2.63}$$



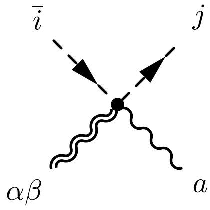
$$= -i\kappa g\Big[P^{\alpha\beta\mu_a\mu_b}\, p_a\cdot p_b + \eta^{\mu_a\mu_b} p_b^{(\alpha} p_a^{\beta)} - \eta^{\mu_a(\alpha} p_b^{\beta)} p_a^{\mu_b} - \eta^{\mu_b(\alpha} p_a^{\beta)} p_b^{\mu_a} + \frac{1}{2}\eta^{\alpha\beta} p_a^{\mu_a} p_b^{\mu_b}\Big] \ , \tag{2.64}$$



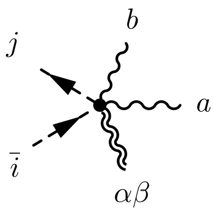
$$\begin{aligned} = \kappa g f^{abc}\Big[& P^{\alpha\beta\mu_a\mu_b}(p_b-p_a)^{\mu_c} + P^{\alpha\beta\mu_a\mu_c}(p_a-p_c)^{\mu_b} \\ &+ P^{\alpha\beta\mu_b\mu_c}(p_c-p_b)^{\mu_a} + \eta^{\mu_a\mu_b}\eta^{\mu_c(\alpha}(p_b-p_a)^{\beta)} \\ &+ \eta^{\mu_a\mu_c}\eta^{\mu_b(\alpha}(p_a-p_c)^{\beta)} + \eta^{\mu_b\mu_c}\eta^{\mu_a(\alpha}(p_c-p_b)^{\beta)}\Big] \ , \end{aligned} \tag{2.65}$$



$$= i\kappa\left[\frac{1}{2}\eta^{\alpha\beta}\left(-p_i\cdot p_j - m_i^2\right) + p_i^{(\alpha} p_j^{\beta)}\right]\delta^{j\bar{i}} \ , \tag{2.66}$$



$$= i\kappa g\left[\frac{1}{2}\eta^{\alpha\beta}\left(p_i-p_j\right)^{\mu_a} - \eta^{\mu_a(\alpha}(p_i-p_j)^{\beta)}\right](T^a)_{j\bar{i}} \ , \tag{2.67}$$



$$= -i\kappa g^2 P^{\alpha\beta\mu_a\mu_b}\{T^a, T^b\} \ , \tag{2.68}$$

where we use notations previously introduced in (2.57). Higher vertices can be found in [77].

In the case of QCD we use the Vielbein formalism in order to expose the explicit geometrical dependence, i.e.

$$\begin{aligned} \mathcal{L}_{\mathrm{QCD}} =& \ \bar{\psi}\left(i\not{D} - m_\psi\right)\psi \\ \rightarrow& \sqrt{-g}\bar{\psi}\left[i\gamma^a\, e_a^\mu\left(\partial_\mu - \frac{i}{2}\, S_{ab}\,\omega_\mu^{ab} - ig\, A_\mu\right) - m_\psi\right]\psi \ , \end{aligned} \tag{2.69}$$

where latin indices are flat and greek are curved. We have introduced the Vielbeins e_a^μ with inverses e_μ^a, the spinor representation of the Lorentz algebra $S_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$ and the spin connection

$$\omega_\mu^{ab} = \frac{1}{2}\, e_\nu^a\, \partial_\mu e^{b\nu} + \frac{1}{2}\, e^{a\nu}\, e^{b\sigma}\, \partial_\sigma\, g_{\mu\nu} - (a \leftrightarrow b)\ . \tag{2.70}$$

From

$$e_\mu^a\, e_\nu^b\, \eta_{ab} = g_{\mu\nu}\ , \tag{2.71}$$

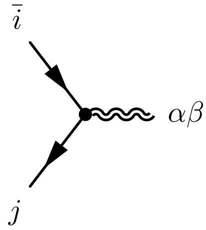
we can deduce the graviton expansion to lowest order

$$\begin{aligned} e_\mu^a &= \delta_\mu^a + \frac{\kappa}{2}\, h_\mu^a + \mathcal{O}(\kappa^2) \\ e_a^\mu &= \delta_a^\mu - \frac{\kappa}{2}\, h_a^\mu + \mathcal{O}(\kappa^2)\ , \end{aligned} \tag{2.72}$$

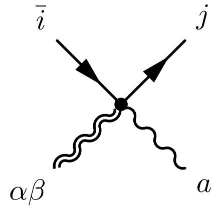
thus obtaining the Einstein-QCD (EQCD) Lagrangian

$$\mathcal{L}_{\text{EQCD}} = \mathcal{L}_{\text{QCD}} + \frac{\kappa}{2}\bar{\psi}\left[h\,(i\not{D} - m_\psi) - i\not{h}^\mu\, D_\mu - 2\,(\partial_a\, \not{h}_b)\, S^{ab}\right]\psi + \mathcal{O}(\kappa^2)\ , \tag{2.73}$$

yielding the one graviton Feynman rules



$$= \frac{i\kappa}{2}\left[\eta^{\alpha\beta}\left(\not{p}_i - m_\psi + \frac{1}{2}\not{p}\right) - \gamma^{(\alpha}(p_i + \frac{1}{2}p)^{\beta)}\right]\delta^{j\bar{i}}\ , \tag{2.74}$$



$$= \frac{i\kappa g}{2}\left[\eta^{\alpha\beta}\gamma^{\mu_a} - \eta^{\mu_a(\alpha}\gamma^{\beta)}\right](T^a)_{j\bar{i}}\ . \tag{2.75}$$

Regarding the previously discussed proper color decompositions not much is changing. Gravitons do not participate in any ordering procedures thus we can proceed in the same fashion as we would for lower point amplitudes without gravitons. Therefore all decompositions generalize trivially and we write

$$\mathcal{A}_{n,m}^{\text{EYM}} = \sum_{\sigma \in S_{n-2}} C_{\text{KK/DDM}}(1, \sigma, n)\mathrm{A}^{\text{EYM}}(1, \sigma, n; h_1, h_2, ..., h_m)\ , \tag{2.76}$$

for a proper decomposition of EYM amplitude with n gluons and m gravitons, which we separate in the amplitude argument using a semicolon, in either the KK or DDM basis. Equivalently we write

$$\mathcal{A}_{n,k,m}^{\text{EsQCD}} = \sum_{\sigma \in \text{MJO basis}}^{\chi(n,k)} C(\underline{1}, \bar{2}, \sigma)\, \mathrm{A}^{\text{EsQCD}}(\underline{1}, \bar{2}, \sigma; h_1, ..., h_m)\ , \tag{2.77}$$

for a scattering with $n-2k$ gluons k fundamental/anti-fundamental pairs and m gravitons.

Ward identities are again satisfied for every gluon and graviton where the shifted particle does not have to be on-shell. Interestingly, neither amplitudes satisfy BCJ relations.

Note that we only consider scattering amplitudes with no graviton propagators connecting colored particles. In the case of EYM amplitudes this is sometimes referred to as single trace sector.

2.5 Universal Properties of Gauge and Gravity Amplitudes

Scattering amplitudes possess very important and powerful factorization properties, controlled by the S-matrix unitarity, which are of critical importance for infrared finiteness of physical observables [52, 53]. In this section we want to review the two arguably most important factorizations, tree level single soft- and collinear gluon and graviton limits [78–80].

Given any partial tree level amplitude which involves an external gluon with momentum q, we can probe the analytic behavior in the limit $q \to \tau\, q$ and $\tau \to 0$. This limit is called single soft gluon limit and the analytic structure of the amplitude is [79–81]

$$\lim_{\tau\to 0} \mathrm{A}_n(..., a, \tau\, q, a+1, ...) = g\left(\frac{1}{\tau} S^{(0)}_{\mathrm{YM}} + S^{(1)}_{\mathrm{YM}}\right) \mathrm{A}_{n-1}(..., a, a+1, ...) + \mathcal{O}(\tau)\ , \tag{2.78}$$

where we observe a factorization into a lower point amplitude and prefactors to which we will refer to as YM soft theorems

$$S^{(0)}_{\mathrm{YM}} = \frac{(\epsilon_q\cdot p_{a+1})}{q\cdot p_{a+1}} - \frac{(\epsilon_q\cdot p_a)}{q\cdot p_a}\quad ,\quad S^{(1)}_{\mathrm{YM}} = \frac{\epsilon^\mu_q\, q^\nu\, J^{a+1}_{\mu\nu}}{q\cdot p_{a+1}} - \frac{\epsilon^\mu_q\, q^\nu\, J^{a}_{\mu\nu}}{q\cdot p_a}\ , \tag{2.79}$$

where

$$J^{\mu\nu}_a = \epsilon^\mu_a\, \partial^\nu_{\epsilon_a} + p^\mu_a\, \partial^\nu_{p_a} - (\mu \leftrightarrow \nu)\ . \tag{2.80}$$

Above expansion is universal, i.e. independent of the other particles properties and the soft factors are obviously gauge invariant. Nevertheless, note that the soft theorems depend on the position of the gluon in the amplitude.

A similar expansion occurs for gravitons. We perform the same single soft limit and obtain

$$\lim_{\tau\to 0} M_n(..., a, \tau\, q, a+1, ...) = \kappa\left(\frac{1}{\tau} S^{(0)}_{\mathrm{QGR}} + S^{(1)}_{\mathrm{QGR}} + \tau S^{(2)}_{wQGR}\right) M_{n-1}(..., a, a+1, ...) + \mathcal{O}(\tau^2)\ , \tag{2.81}$$

with the gauge invariant soft theorems

$$\begin{aligned}
S^{(0)}_{\mathrm{QGR}} &= \sum_{a=1}^{n-1} \frac{(\epsilon_q\cdot p_a)^2}{q\cdot p_a}\ ,\\
S^{(1)}_{\mathrm{QGR}} &= \sum_{a=1}^{n-1} \frac{(\epsilon_q\cdot p_a)\,(\epsilon^\mu_q\, q^\nu\, J^a_{\mu\nu})}{q\cdot p_a}\ ,\\
S^{(2)}_{\mathrm{QGR}} &= \sum_{a=1}^{n-1} \frac{(\epsilon^\mu_q\, q^\nu\, J^a_{\mu\nu})^2}{q\cdot p_a}\ .
\end{aligned} \tag{2.82}$$

Again, this expansion is universal [82] and, in this case, independent of the gravitons position.

Soft limits have gotten a lot of attention in the past due to very intricate connections to hidden symmetries and spontaneous symmetry breaking [83–87]. Very recently is has been discovered that locality together with the gluon and graviton leading single soft theorems or gauge invariance uniquely fixes the full amplitude in the case of YM and QGR amplitudes [88] by constraining a generic polynomial ansatz. Thus any local function which satisfies the leading gluon soft theorem or gauge invariant in every particle must be the associated partial YM amplitude. The same is true for QGR amplitudes and for mixed amplitudes of gravitons and gluons. We will use this observations as a guiding principle fortifying intuitive constructions of amplitude connections in later chapters.

Regarding collinear limits, much less is known. A collinear limit is defined by a kinematic configuration where two arbitrary momenta in the scattering process become proportional

$$p_i = x\,p \quad , \quad p_j = (1-x)\,p \quad , \quad \to p_i + p_j = p \quad , \tag{2.83}$$

where $x \leq 1$ is called momentum fraction and the on-shell momentum p will be called fused momentum. We will compactly refer to such a kinematic arrangement as $p_i||p_j$. In this limit the kinematic space naturally reduces by one particle.

For collinear gluons in a tree level YM partial amplitude two kinds of behaviors can occur. Either the two collinear gluons are adjacent or not. In the adjacent case the amplitude factorizes universally in the following way [89]

$$A_n(1^{h_1}, 2^{h_2}, \ldots) \xrightarrow{1\|2} \underbrace{\sum_{h=\pm} \text{Split}_{-h}(x; 1^{h_1}, 2^{h_2})}_{\propto \frac{1}{\epsilon}} \; A_{n-1}(p^h, \ldots) + \mathcal{O}(1) \ , \tag{2.84}$$

where ϵ is a parameter controlling the divergence and the function $\text{Split}_{-h}(x; 1^{h_1}, 2^{h_2})$ is called Splitting function (also known as the Altarelli-Parisi factor) which encapsulates the divergent structure, i.e. the three gluon vertex. Note that the helicity dependence of the collinear gluons is exposed explicitly in above notation and no position dependence of the two gluons is present.

In the non-adjacent case not much is known except that the amplitude is not factorizing and no divergences are present.

In the case of collinear gravitons also very few properties have been studied and established. This is due to the fact that graviton amplitudes do not encode any particle ordering dependence and thus no adjacent collinear limit can be formulated, excluding divergences. Nevertheless, it is known that the leading collinear limit can be written as a sum of a factorized part with universal behavior and something else which is neither known to be universal nor factorizable [90, 91].

Collinear limits will be of great interest in future chapters due to intricate connections between adjacent collinear gluon limits and EYM amplitudes which we will use as a starting point. Along the way we will fill relevant gaps in our understanding of collinear limits.

2.6 The Yang-Mills Double Copy

A highly remarkable result of the last decade is the observation that various QFTs are connected at the level of amplitudes [23–25, 92]. In this section we want to present one of such connections between YM and gravity amplitudes referred to as the YM double copy [11–13, 93]. Note that by gravity we do not necessarily mean QGR. In fact, the double copy does not straightforwardly involve QGR except at tree level. Still, gravity always refers to a theory of a self interacting, massless spin 2 boson. Before getting into detail about what we mean by gravity, let us explain the double copy.

We stress that it is possible to dive very deep into this topic since it has been subject to a great amount of very successful research [14–22, 28]. We will therefore restrict ourselves to the basic concepts and ideas which we will use as guiding principles in later chapters. We start by noting that scattering amplitudes at any loop order in in d spacetime dimensions in YM and gravity theory are always of the schematic form

$$\begin{aligned}\mathcal{A}_n^{\mathrm{YM},(L)} &= i^{L-1}\,\mathrm{g}^{n-2+2L} \sum_{i\in\mathrm{graphs}} \int \prod_{a=1}^{L} d^d\, l_a\, \frac{1}{\mathrm{S}_i}\, \frac{\mathrm{C}_i\,\mathrm{N}_i}{\mathrm{D}_i} \quad , \\ \mathcal{M}_n^{\mathrm{G},(L)} &= i^{L-1} \left(\frac{\kappa}{2}\right)^{n-2+2\ L} \sum_{i\in\mathrm{graphs}} \int \prod_{a=1}^{L} d^d\, l_a\, \frac{1}{\mathrm{S}_i}\, \frac{\tilde{\mathrm{N}}_i}{\mathrm{D}_i}\ ,\end{aligned} \tag{2.85}$$

as consequence of Feynman rules. Here L is the loop level, the superscript G indicates some gravity theory and we extracted factors of i, g, κ according to the double copy convention. We keep the standard definitions from (2.9), (2.60) for C_i, D_i, N_i and $\tilde{N}_i$. Finally, we denote S_i as the symmetry factor for single loop level graphs.

At this point it is necessary to perform two manipulations to the YM amplitude.

First, we have to collect terms in the YM amplitude in such a way that we can restrict the sum over all graphs to a set of cubic graphs, i.e. Feynman graphs built from three vertices only. Clearly, the four gluon vertex poses an obstacle. The tension is relieved by associating the contribution of every four gluon vertex in a graph to the same graph topology except that the four gluon vertex is replaced by a sum of s-, t- and u-channel configurations. More explicitly, we realize that the color structure of the four gluon vertex is the sum of s-, t- and u-channel color structures weighted by sums of products of Minkowski metrics (2.7). A minor adjustment is necessary, since the four gluon vertex will still miss the propagator structure of a cubic diagram. This can easily be corrected by multiplying and dividing the individual terms in the four gluon vertex by the appropriate Mandelstam invariants of the s-, t-, and u-channel. The last step is to combine all terms by redefining the YM numerator structures such that the sum over all graphs only involves cubic diagrams. This procedure is called decomposition of the four gluon vertex.

Let us illustrate this procedure for the all incoming four gluon amplitude

$$\mathcal{A}_4 = \mathrm{g}^2 \left(\frac{\mathrm{C}_s\,\mathrm{N}_s}{\mathrm{D}_s} + \frac{\mathrm{C}_t\,\mathrm{N}_t}{\mathrm{D}_t} + \frac{\mathrm{C}_u\,\mathrm{N}_u}{\mathrm{D}_u} + \varepsilon_1^{\mu_1}\,\varepsilon_2^{\mu_2}\,\varepsilon_3^{\mu_3}\,\varepsilon_4^{\mu_4}\,\mathrm{V}^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4} \right) \ , \tag{2.86}$$

with the intuitive assignments

$$
\begin{aligned}
&\mathrm{C}_s = -2\, f^{a_1 a_2 b}\, f^{b a_3 a_4} \quad , \quad \mathrm{C}_t = -2\, f^{a_2 a_3 b}\, f^{b a_1 a_4} \quad , \quad \mathrm{C}_u = -2\, f^{a_3 a_1 b}\, f^{b a_2 a_4} \quad , \\
&\mathrm{N}_s = \frac{i}{2}\, [(\varepsilon_1 \cdot \varepsilon_2)\; p_1^\mu + 2\,(\varepsilon_1 \cdot p_2)\; \varepsilon_2^\mu - (1 \leftrightarrow 2)]\; [(\varepsilon_3 \cdot \varepsilon_4)\; p_{3,\mu} + 2\,(\varepsilon_3 \cdot p_4)\; \varepsilon_{4,\mu} - (3 \leftrightarrow 4)] \quad , \\
&\mathrm{N}_u = \mathrm{N}_s|_{1\to 3\to 2\to 1} \quad , \quad \mathrm{N}_t = \mathrm{N}_s|_{1\to 2\to 3\to 1} \quad , \\
&\mathrm{D}_s = (p_1 + p_2)^2 \quad , \quad \mathrm{D}_t = (p_1 + p_4)^2 \quad , \quad \mathrm{D}_u = (p_1 + p_3)^2 \quad .
\end{aligned}
\tag{2.87}
$$

In above expression on-shell conditions $\varepsilon_i \cdot p_i = p_i^2 = 0$ have been used and the prefactor assignments are chosen such that they mimic the scattering amplitudes literature conventions. The last term in $\mathcal{A}_4$ is the four gluon vertex contracted with gluon polarizations, i.e.

$$
\varepsilon_1^{\mu_1}\, \varepsilon_2^{\mu_2}\, \varepsilon_3^{\mu_3}\, \varepsilon_4^{\mu_4}\, \mathrm{V}^{a_1 a_2 a_3 a_4}_{\mu_1 \mu_2 \mu_3 \mu_4} = C_s\, \hat{N}_s + C_t\, \hat{N}_t + C_u\, \hat{N}_u \ , \tag{2.88}
$$

where

$$
\begin{aligned}
\hat{N}_s &= \frac{i}{2}\, [(\varepsilon_1 \cdot \varepsilon_3)\, (\varepsilon_2 \cdot \varepsilon_4) - (\varepsilon_1 \cdot \varepsilon_4)\, (\varepsilon_2 \cdot \varepsilon_3)] \quad , \\
\hat{N}_u &= \hat{N}_s|_{1\to 3\to 2\to 1} \quad , \quad \hat{N}_t = \hat{N}_s|_{1\to 2\to 3\to 1} \quad .
\end{aligned}
\tag{2.89}
$$

Clearly, we can define new cubic numerators

$$
n_s = \mathrm{N}_s + \mathrm{D}_s\, \hat{N}_s \quad , \quad n_t = \mathrm{N}_t + \mathrm{D}_t\, \hat{N}_t \quad , \quad n_u = \mathrm{N}_u + \mathrm{D}_u\, \hat{N}_u \quad , \tag{2.90}
$$

which are associated to cubic diagrams and include the four gluon vertex contributions. Hence, we can write

$$
\mathcal{A}_4 = \mathrm{g}^2 \left(\frac{C_s\, n_s}{D_s} + \frac{C_t\, n_t}{D_t} + \frac{C_u\, n_u}{D_u} \right) = \mathrm{g}^2 \sum_{i \in \text{cubic graphs}} \frac{C_i\, n_i}{D_i} \ , \tag{2.91}
$$

arriving at the desired cubic representation of $\mathcal{A}_4$. It is clear, that this procedure generalizes to arbitrary point- and loop-number of the YM amplitude.

The second manipulation of the YM amplitude concerns gauge invariance. Replacing any polarization by its momentum has to nullify the full amplitude. Nevertheless, single Feynman diagrams and thus kinematic numerators are not gauge invariant. We can see this explicitly from our four gluon example

$$
n_s|_{\varepsilon_4 \to p_4} = i\, \frac{\mathrm{D}_s}{2}\, [(\varepsilon_1 \cdot \varepsilon_2)\; (\varepsilon_3 \cdot (p_2 - p_1)) + (1 \to 2 \to 3 \to 1) + (1 \to 3 \to 2 \to 1)] = \mathrm{D}_s\, \xi \ , \tag{2.92}
$$

where ξ is defined intuitively. The result is non-zero and we may ask how gauge invariance is manifested in the full amplitude. To do so, we sum up all contributions while using $n_t|_{\varepsilon_4 \to p_4} = \mathrm{D}_t\, \xi$, $n_u|_{\varepsilon_4 \to p_4} = \mathrm{D}_u\, \xi$

$$
\frac{\mathrm{C}_s\, n_s|_{\varepsilon_4 \to p_4}}{\mathrm{D}_s} + \frac{\mathrm{C}_t\, n_t|_{\varepsilon_4 \to p_4}}{\mathrm{D}_t} + \frac{\mathrm{C}_u\, n_u|_{\varepsilon_4 \to p_4}}{\mathrm{D}_u} = (\mathrm{C}_s + \mathrm{C}_t + \mathrm{C}_u)\; \xi \ , \tag{2.93}
$$

which vanishes on the support of Lie-algebra Jacobi relations

$$f^{a_1 a_2 b} f^{b a_3 a_4} + f^{a_2 a_3 b} f^{b a_1 a_4} + f^{a_3 a_1 b} f^{b a_2 a_4} = 0 \ . \tag{2.94}$$

We conclude that gauge invariance is a result of the underlying Lie-algebra structure of the gauge group. In particular, it seems that the only relevant algebra property is the Jacobi relation and thus gauge invariance is preserved even if we replace the color structures by arbitrary functions of the external kinematic data as long as they satisfy Jacobi relations. It is highly remarkable that the we can show with a bit of algebra that

$$n_s + n_t + n_u = 0 \ , \tag{2.95}$$

on the support of momentum conservation and on-shell conditions. Thus we may just replace the color structures by the kinematic structures and obtain

$$\frac{n_s\, n_s}{\mathrm{D}_s} + \frac{n_t\, n_t}{\mathrm{D}_t} + \frac{n_u\, n_u}{\mathrm{D}_u} \ , \tag{2.96}$$

with gauge invariance in both factors of kinematic numerators. We may ask what kind of object we have obtained by our replacement. It is a function with a cubic denominator structure, two powers of polarization for every momentum and and doubled dimensionality, all being properties of a cubic decomposition of a four point amplitude with external spin 2 gauge bosons. Indeed, we find the incredible identification

$$\mathcal{M}_4^{\mathrm{QGR}} = \frac{\kappa^2}{4} \left(\frac{n_s\, n_s}{\mathrm{D}_s} + \frac{n_t\, n_t}{\mathrm{D}_t} + \frac{n_u\, n_u}{\mathrm{D}_u} \right) \ , \tag{2.97}$$

where we now explicitly refer to scattering amplitudes in QGR, i.e. four graviton scattering. This is called the YM double copy procedure.

Let us summarize and give the general procedure and conjecture. Given any YM amplitude we first read through every graph and decompose all four gluon vertices to obtain a cubic decomposition

$$\mathcal{A}_n^{\mathrm{YM},(L)} = i^{L-1}\, \mathrm{g}^{n-2+2L} \sum_{i \in \text{cubic graphs}} \int \prod_{a=1}^{L} d^d\, l_a\, \frac{1}{\mathrm{S}_i} \frac{\mathrm{C}_i\, n_i}{\mathrm{D}_i} \ . \tag{2.98}$$

Note that now each symmetry factor is associated with cubic graphs. This has to be accounted for in the decomposition of a four gluon vertex by multiplying and dividing by the cubic symmetry factor. Afterwards, we identify triples of color structures C_i which, schematically, satisfy

$$\mathrm{C}_i \pm \mathrm{C}_j \pm \mathrm{C}_k = 0 \ , \tag{2.99}$$

on the support of Jacobi relations. Finally, it is necessary to find a representation of the cubic numerators that multiply the individual color structures of above triple such that they satisfy the same Jacobi-like relation

$$n_i \pm n_j \pm n_k = 0 \ , \tag{2.100}$$

which is, in general, never guaranteed and was manifest by accident in the four point example. This is the most crucial part of the double copy procedure and is called color-kinematics duality. We call a triple of numerators satisfying Jacobi relations color dual. If a color dual representation of the numerators is found, the gravity amplitude is given by the simple replacement rule

$$\begin{aligned}
\mathcal{M}_n^{\mathrm{G},(L)} &= \mathcal{A}_n^{\mathrm{YM},(L)}\Big|_{\mathrm{C}_i\to n_i\ ,\ \mathrm{g}\to\frac{\kappa}{2}} \\
&= i^{L-1}\,\mathrm{g}^{n-2+2L}\sum_{i\in\text{cubic graphs}}\int\prod_{a=1}^{L}d^d\,l_a\,\frac{1}{\mathrm{S}_i}\,\frac{\mathrm{C}_i\,n_i}{\mathrm{D}_i}\Big|_{\mathrm{C}_i\to n_i\ ,\ \mathrm{g}\to\frac{\kappa}{2}} \\
&= i^{L-1}\left(\frac{\kappa}{2}\right)^{n-2+2L}\sum_{i\in\text{cubic graphs}}\int\prod_{a=1}^{L}d^d\,l_a\,\frac{1}{\mathrm{S}_i}\,\frac{n_i\,n_i}{\mathrm{D}_i}\ .
\end{aligned} \tag{2.101}$$

Two important points have to be discussed. First, we now again refer to some gravity theory and not QGR as we did in the four point example. The double copy was originally observed and conjectured at any loop level for supergravity amplitudes and it was observed that supersymmetry could be completely removed without changing the double copy logic. Unfortunately, supergravity without supersymmetry is not directly QGR but QGR, i.e. gravitons, augmented by a real scalar field, the dilaton, and an antisymmetric spin 2 boson, the axion. Considering amplitudes with only external gravitons, it is easy to show that axions and dilatons can only appear at loop level due to the absence of linear axion/dilaton terms (the lowest order is given by their kinetic terms which are bilinear).

Second, it is highly non-trivial to find color dual numerators. In fact, the ability to find them is only proven at tree level [13, 13, 94] and even though a tremendous amount of evidence exists [28], the existence of color dual numerators is conjectural. We want to discuss one way of finding them very briefly. One can use so called generalized gauge transformations for a triple of numerators, whose color coefficients satisfy Jacobi relations, and then shift all three numerators by the same function of momenta and polarizations

$$n_i \to n_i + \Delta\ \ ,\ \ n_j \to n_j + \Delta\ \ ,\ \ n_k \to n_k + \Delta \quad \text{while} \quad c_i \pm c_j \pm c_k = 0\ . \tag{2.102}$$

Such a shift will not affect the full amplitude due to color Jacobi relations but the sum of the three numerators is now modified and potentially zero.

We conclude that the double copy procedure maps gluon amplitudes to graviton amplitudes with internal dilatons and axions at loop level and directly to gravitons at tree level. We want to stress that on-shell conditions and gauge invariance play a crucial role in the double copy construction. We will see the ideas and concepts of this section in later chapters.

2.7 Stieberger-Taylor Relations

Apart from the double copy, other relations between gravity and YM amplitudes have been introduced recently by S.Stieberger an T.Taylor [38, 39]. They show how in certain situations single external gravitons in partial EYM amplitudes can be replaced by gluons. The derivation is based on the field theoretic limit of string theory amplitudes. Their first work concerns the collinear limit of two equal helicity gluons in a YM partial amplitude where the finite contribution is associated, through a specific linear combination, to an EYM partial amplitude where the two collinear gluons are replaced by a graviton.

The first formula concerns the adjacent collinear case. Let us call it ST-one

$$\frac{\kappa\, x}{\mathrm{g}^2}\sum_{\rho\in S_{n-3}}(-1)^{m_\rho}\, s_{\rho(n-2)p}\, \mathrm{A}_n^{\mathrm{YM}}(1,\rho(2),...,\rho(n-2),n-1,n) \;=\mathrm{A}_{n-1}^{\mathrm{EYM}}(1,....,n-2;p)\;, \tag{2.103}$$

where $p_n \| p_{n-1}$ with fused momentum p, $m_\rho=\{0,1\}$ for even and odd numbers of permutations ρ respectively, x is the momentum fraction, g and κ are the couplings of gluons and gravitons respectively and $s_{ij}=2\,p_i\,p_j$ are Mandelstam invariants. ST-one seems to be pathological for $x\to 0$, i.e. the single gluon soft limit on one of the collinear gluons, since it implies that the EYM amplitude vanishes. We will see that this is not the case since the LHS of ST-one will be $\propto \frac{1}{x}$. Note that the leading collinear contribution cancels out because of BCJ relations (2.19). More explicitly, consider the four point case

$$s_{3p}\,\mathrm{A}(1,2,3,4,5)-s_{2p}\mathrm{A}(1,3,2,4,5)=\mathrm{Split}(4,5)\;(s_{3p}\mathrm{A}(1,2,3,p)-s_{2P}\mathrm{A}(1,3,2,p))+...\;, \tag{2.104}$$

where the dots denote higher orders. We can massage the Mandelstam invariants using momentum conservation and on-shell conditions

$$s_{3p}=-s_{32}-s_{31}\quad,\quad s_{2p}=s_{31}\quad, \tag{2.105}$$

yielding

$$(s_{31}+s_{31})\mathrm{A}(1,2,3,p)+s_{31}\mathrm{A}(1,3,2,p)=2\,p_3\,X_3\,\mathrm{A}(1,2,3,p)+2\,p_3\,X_3\,\mathrm{A}(1,3,2,p)\quad, \tag{2.106}$$

which is exactly a BCJ relation for gluon with momentum p_3 and therefore vanishing.

It is rather counter intuitive from first looking upon (2.103) that BCJ relations still exist among the partial amplitudes since three legs have been fixed in their position, i.e. leg 1 and the two collinear legs, already requiring the use of all amplitude relations. The crux here is that by switching to a collinear configuration for two of the fixed legs effectively leave the basis, since two individual BCJ relations become equal in collinear kinematics. We can also argue that since collinear kinematics fuse two particles together we should always regard the two collinear legs as one label which implies that (2.103) is actually in the KK/DDM basis. This on the other hand implies that BCJ relations still exists. In fact, we will see that ST-one can be interpreted as a deviation from BCJ relations due to collinear kinematics, constraining a possible factorization of the sub-leading collinear limit. We will make this point explicit in future chapters.

The second relation in [38] concerns the non-adjacent collinear limit of two gluons in a tree level YM amplitudes. Again, a connection to an EYM tree level amplitude, with a graviton replacing the two gluons, is given. A very similar logic to ST-one but with very different technical execution. The generic formula is rather involved and we will focus on an 5-point example

$$\mathrm{A}^{\mathrm{EYM}}(1,2,3,4;p)=\frac{\kappa\,(1-x)}{\mathrm{g}^2}\,[s_{25}\mathrm{A}(1,6,2,5,3,4)+s_{45}\mathrm{A}(1,2,3,5,4,6)]\Big|_{5\|6}\quad. \tag{2.107}$$

We will call such relations ST-two. In this case the YM partial amplitudes have no relations used among them, i.e. only leg 1 is fixed. This is due to the fact, that only non-adjacent collinear configurations are considered and KK- as well as BCJ-relations move both legs adjacent. We will not dive deeper into this identity but rather focus on ST-one.

The last relation is presented in [39] and follows from ST-one through a single soft limit on one of the collinear constituents, i.e. either $x \to 0$ or $x \to 1$. In this limit the sub-leading collinear limit factorizes and ST-one transforms into

$$\sum_{\sqcup\!\sqcup} \varepsilon_p \cdot X_p \, \mathrm{A}^{\mathrm{YM}}(1, 2, \{3, ..., n\} \sqcup\!\sqcup \{p\}) = \frac{\kappa}{2\mathrm{g}^2} \, \mathrm{A}^{\mathrm{EYM}}(1, ..., n; p) \ , \tag{2.108}$$

where we use the notion of a region momentum X_p defined in (2.20) and ε_p is the fused polarization vector of the two collinear momenta. We call this relation ST-three. The YM amplitudes on the LHS are in the KK or DDM basis and no residual information of collinear kinematics is left over. Thus ST-three is an exact statement in all kinematic regions. The similarity to BCJ relations is striking and in fact we can interpret BCJ relations as the gauge invariance condition on an EYM amplitude, i.e. replacing the graviton polarization by the associated momentum will also involve a replacement in amplitudes prefactors on the LHS reproducing BCJ relations exactly. Note that this interpretation is different to ST-one which we understood as a deviation from BCJ relations without any connection to gauge invariance at first glance.

In order to avaoid any confusion, we want to stress that no internal gravitons are present in all above formulae.

We end this chapter noting a key difference between the YM double copy and ST relations. While the double copy refrains from any color decomposition and requires a very specific set up using color kinematics duality, ST relations do not involve any color at all and work at the level of partial amplitudes. We will explore this discrepancy in an upcoming chapter with the specific question whether both types of gluon/graviton connection can be brought on equal footing.

2.8 The Cachazo-He-Yuan Formalism

Arguably one of the biggest disadvantages in computing scattering amplitudes with Feynman diagrams is the size of intermediate expressions, especially when dealing with non-abelian gauge or gravity theories. It is therefore a remarkable result that there exists a formula yielding compact expressions for tree level scattering amplitudes in a wide variety of theories and in arbitrary spacetime dimensions. This formula is called Cachazo-He-Yuan formalism (CHY). It was proposed in [48–50, 95, 96] and proven shortly after in [97]. In the, so-called, CHY frame, tree level scattering amplitudes with massless external bosons are expressed through a fully localized intergral over a punctured Riemann sphere $\mathbb{C}\mathrm{P}^1$. The formula reads

$$\mathrm{A}_n^{\mathrm{Th}}(\{p, \varepsilon\}) = \int_{\mathbb{C}\mathrm{P}^1} d\mu_n \, \mathcal{I}_n^{\mathrm{Th}}(\{p, \varepsilon, \sigma\}) \ , \tag{2.109}$$

where p, ε is the set of all external momenta and polarizations, i.e. $\{p, \varepsilon\} = \{p_1, \varepsilon_1, ..., p_n, \varepsilon_n\}$ and the superscript "Th" denotes a theory sensitivity. Note that, traditionally all coupling constants are neglected. Let us dissect above formula. Each external particle with data $\{p_i, \varepsilon_i\}$ is associated with a puncture σ_i on $\mathbb{C}\mathrm{P}^1$. The integral is then taken over the punctures with measure

$$d\,\mu_n = d'\sigma_n\,\Delta'_n$$

$$d'\sigma_n = (\sigma_{ij}\,\sigma_{jk}\,\sigma_{ki}) \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} d\sigma_a \quad , \quad \Delta'_n = (\sigma_{pq}\,\sigma_{qr}\,\sigma_{rp}) \prod_{\substack{b=1\\ b\neq p,q,r}}^{n} \delta(f_b)\ , \tag{2.110}$$

where we introduced the shorthand $\sigma_{ab} = \sigma_a - \sigma_b = -\sigma_{ba}$ and the objects f_a

$$f_a = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{2\,p_a\cdot p_b}{\sigma_a - \sigma_b}\ , \tag{2.111}$$

which enforce the, highly gripping, scattering equations

$$f_a = 0\ , \tag{2.112}$$

through a delta function constraint, thus relating momenta and punctures in a non-linear way. Scattering equations have a long history in a wide range of topics [98–104]. From the definition of the measure we learn that the integral is only performed for $n-3$ punctures. The remaining punctures appear in a specific product combination. Equivalently, only $n-3$ scattering equations appear as delta constraints thus fully localizing the integral. This property originates from a gauge fixing of $\mathrm{SL}(2,\mathbb{C})$ symmetry on $\mathbb{C}\mathrm{P}^1$ which we shall call modular gauge fixing. We do not need to go into much detail regarding modular symmetry[5] but we want to summarize two important results. The choice of indices i, j, k and p, q, r is arbitrary. The measure is permutation invariant and the CHY integral is localized on $(n-3)!$ solutions [48]. The measure does not carry a "Th" index indicating universality among all theories.

The object $\mathcal{I}_n^{\mathrm{Th}}(\{p, \varepsilon, \sigma\})$ in (2.109) is called the CHY-integrand and it determines the theory origin of the amplitudes. The integrand is always a product of so-called CHY building blocks, two of which are of special interest in this thesis. The first building block is the Parke-Taylor (PT) factor, which has a familiar structure when comparing it to a YM MHV amplitude developed by Parke and Taylor in [61], i.e.

$$\mathfrak{C}(1, ..., n) = \frac{1}{\sigma_{12}\sigma_{23}\cdots\sigma_{n-1,n}\sigma_{n1}}\ . \tag{2.113}$$

The other building block involves an antisymmetric $2n\times 2n$ matrix Ψ_n defined by

$$\Psi_n = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}\ , \tag{2.114}$$

where the entries are $n\times n$ matrices given by

$$A_{ab} = \begin{cases} \frac{2p_a\cdot p_b}{\sigma_a-\sigma_b} & a\neq b\,, \\ 0 & a=b\,, \end{cases} \qquad B_{ab}^{h_a|h_b} = \begin{cases} \frac{2\varepsilon_a^{h_a}\cdot\varepsilon_b^{h_b}}{\sigma_a-\sigma_b} & a\neq b\,, \\ 0 & a=b\,, \end{cases} \qquad C_{ab}^{h_a} = \begin{cases} \frac{2\varepsilon_a^{h_a}\cdot p_b}{\sigma_a-\sigma_b} & a\neq b\,, \\ -\sum\limits_{c\neq a} \frac{2\varepsilon_a^{h_a}\cdot p_c}{\sigma_a-\sigma_c} & a=b\,. \end{cases} \tag{2.115}$$

[5] The interested reader is referred to [105]

Here h_i denotes the helicity carried by the massless boson. The building block obtained from this matrix is Either a Pfaffian or a reduced Pfaffian. We separate the notation through a prime superscript similar to the CHY measure (2.110), i.e.

$$\textbf{unreduced:} \quad \text{Pf } \Psi_m \quad , \quad \textbf{reduced:} \quad \text{Pf}' \ \Psi_n = \frac{(-1)^{i+j}}{\sigma_{ij}} \text{Pf } \Psi_n^{i,j} \ , \tag{2.116}$$

where the choice of i, j is arbitrary but must be unequal and taken form $\{i, j\} \in \{1, ..., n\}$. The superscript above the Ψ_n matrix in the reduced Pfaffian denotes that both rows and columns i, j are deleted from the matrix. Note that the unreduced Pfaffian can only contain a matrix for a subset of all n external particles, i.e. $m < n$, since the Pfaffian vanishes, on the support of scattering equations, otherwise. In particular, the Pfaffian of an antisymmetric matrix M is defined by

$$\text{Pf } M = \sqrt{\det M} \ . \tag{2.117}$$

Thus, key matrix manipulation techniques are inherited for a Pfaffian computation, i.e. it is possible to add a multiple of a row and its corresponding column to another row and corresponding column without changing the value of the Pfaffian. Therefore, the unreduced Pfaffian of Ψ_n is zero since we can add $n-1$ rows and columns to the remaining row and column, let us say th i'th, in which all arguments will be either f_i in the A matrix of Ψ_n or 0 in the C matrix. Thus, enforcing the scattering equation $f_i = 0$ produces a row and column full of zeros yielding a vanishing Pfaffian. This explains the need for a reduced Pfaffian [48].

Note that the Pfaffian is permutation invariant and any ordering dependence of a scattering amplitude is encoded in the PT factor. For gauge theory amplitudes, gauge invariance is checked remarkably simple in the Pfaffian, since a replacement $\varepsilon_a \to p_a$ will produce two equal rows and columns in Ψ.

The most relevant integrands for YM, QGR and bi-adjoint scalar theory are given by

$$\begin{aligned} \textbf{YM partial amplitudes:} \quad & \mathcal{I}_n^{\text{YM}}(1, ..., n) = \mathfrak{C}(1, ..., n) \ \text{Pf}' \ \Psi_n \ , \\ \textbf{QGR partial amplitudes:} \quad & \mathcal{I}_n^{\text{QGR}}(1, ..., n) = \mathfrak{C}(1, ..., n) \ \text{Pf}' \ \Psi_n \ \text{Pf}' \ \Psi_n \ , \\ \textbf{Bi-adjoint scalar amplitudes:} \quad & \mathcal{I}_n^{\text{Bi-adjointscalar}}(1, ..., n) = \mathfrak{C}(1, ..., n)^2 \ . \end{aligned} \tag{2.118}$$

Note the double-copy-like structure. One of the biggest advantage of the CHY formalism is the ability to easily consider mixed scattering scenarios between gluons, gravitons and scalars, i.e. Einstein-scalar-Yang-Mills (EsYM) amplitudes, simply by including the proper integrand for the particles. To make this point clear, we consider an integrand for such a scattering, i.e.

$$\mathcal{I}_{k+r+m}^{\text{EsYM}}(g_1, ..., g_k, s_1, ..., s_r; h_1, ..., h_m) = \mathfrak{C}(g_1, ..., g_k, s_1, ..., s_r) \, \mathfrak{C}(s_1, ..., s_r) \ \text{Pf } \Psi_m \ \text{Pf}' \ \Psi_{k+m} \ , \tag{2.119}$$

with k gluons, m gravitons and r scalars. Thus, we get a PT factor for all gluons and scalars, one PT factor for all scalars, a unreduced Pfaffian for the gravitons and one reduced Pfaffian for all gluons and gravitons. Note that every graviton and gluon appears with the correct amount of polarization vectors.

We will end this section with a qualitative discussion of the efficiency and practical value of the CHY formalism. We note that in order to obtain a final result, it is necessary to solve all scattering equations (2.112) simultaneously. This is problematic because every puncture σ_i appears in a denominator and thus one has to solve a high degree polynomial for every σ_i in a general setting. A solution quickly becomes unreachable analytically (and also numerically). There exist various proposals to circumvent this issue by using certain algorithms [106–108] but overall no real breakthrough has occurred, thus devalue the CHY formalism to a useful tool for amplitude computations. So why is it that we care about it? The striking advantages are that it is the only formula which captures the analytic form of non-abelian gauge and gravity amplitudes in a compact way up to any number of external legs, making it the perfect playground to test and derive new universal mathematical statements about amplitudes. Furthermore, it is the only formalism that allows to study relations between amplitudes in different theories due to its building block structure. The price to pay is then the lack of efficiency. In the next section we will be introducing basic computational techniques for the CHY formalism which will allow us to derive and proof very powerful statements about gauge and gravity amplitudes with elegant simplicity.

2.9 The Insertion Operator

In this section we want to introduce computational techniques in the CHY frame. We will derive, and thus proof, relations between partial YM amplitudes introduced in section 2.1. A color ordered YM amplitude is given in the CHY frame as

$$\mathrm{A}_n^{\mathrm{YM}}(1,...,n) = \int_{\mathbb{CP}^1} d\mu_n\, \mathfrak{C}(1,...,n)\ \mathrm{Pf}'\,\Psi_n\ . \tag{2.120}$$

Cyclicity and reflectivity are proven trivially, since the Pfaffian and measure are permutation invariant and the explicit structure of the PT factor is reflective and cyclic invariant. In order to prove KK and BCJ relations we introduce the notion of an, very useful, insertion operator

$$S(a,b,c) = \frac{\sigma_{ac}}{\sigma_{ab}\,\sigma_{bc}} = \frac{1}{\sigma_{ab}} + \frac{1}{\sigma_{bc}} = -\,S(c,b,a)\ , \tag{2.121}$$

with a very nice and obvious decomposition property

$$S(a,b,c) = S(a,b,k) + S(k,b,c)\ , \tag{2.122}$$

for some label $k \neq a,b$. From this definition it is trivial to see that any PT factor can be recursively obtained via

$$\mathfrak{C}(1,...,a,b,c,...,n) = S(a,b,c)\,\mathfrak{C}(1,...,a,c,...,n)\ , \tag{2.123}$$

with the straightforward generalization through the iterative use of (2.122), i.e. given some set $B = \{b_1,...,b_m\} \subset \{1,...,n\}$ and a label $k \notin \{1,...,n\}$ we get

$$\begin{aligned} &S(a,k,c)\,\mathfrak{C}(1,...,a,B,c,...,n) \\ &= \left(S(a,k,b_1) + \sum_{i=1}^{m-1} S(b_i,k,b_{i+1}) + S(b_m,k,c)\right)\mathfrak{C}(1,...,a,B,c,...,n) \\ &= \sum_{\sqcup} \mathfrak{C}(1,...,a,\{B\}\sqcup\{k\},c,...,n)\ , \end{aligned} \tag{2.124}$$

with the shuffle product as defined in (2.17).

We can prove KK relations for the PT factor, and thus the amplitude, by considering a specific arrangement of insertion operators, i.e. for some disjoint sets of labels $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$ we can write

$$\begin{aligned}\mathfrak{C}(1, A, k, B) &= \prod_{i=1}^{n-1} S(1, a_i, a_{i+1})\, S(1, a_n, k)\, \mathfrak{C}(1, k, B) \\ &= (-1)^n \prod_{i=1}^{n-1} S(a_{i+1}, a_i, 1)\, S(k, a_n, 1)\, \mathfrak{C}(k, B, 1) \ ,\end{aligned} \tag{2.125}$$

obtained from a repeated use of (2.123) and anti-symmetry in the peripheral labels of the insertion operator (2.121). Using (2.124) repeatedly yields

$$\prod_{i=1}^{n-1} S(a_{i+1}, a_i, 1)\, S(k, a_n, 1)\, \mathfrak{C}(k, B, 1) = \sum_{\sqcup\!\sqcup} \mathfrak{C}(1, k, \{B\} \sqcup\!\sqcup \{a_n, a_{n-1}, a_{n-2}, ..., a_1\}) \ , \tag{2.126}$$

since the insertion of a_i can only occur somewhere in between a_{i+1} and 1, i.e.

$$\begin{aligned}&S(a_n, a_{n-1}, 1) \sum_{\sqcup\!\sqcup} \mathfrak{C}(k, \{B\} \sqcup\!\sqcup \{a_n\}, 1) = \\ &S(a_n, a_{n-1}, 1) \left(\sum_{i=1}^{m-1} \mathfrak{C}(k, ..., b_i, a_n, b_{i+1}, 1) + \mathfrak{C}(k, a_n, B, 1) + \mathfrak{C}(k, B, a_n, 1) \right) = \\ &\quad \sum_{i=1}^{m-1} \mathfrak{C}(k, ..., b_i, a_n, \{b_{i+1}, ..., b_m\} \sqcup\!\sqcup \{a_{n-1}\}, 1) + \mathfrak{C}(k, a_n, \{B\} \sqcup\!\sqcup \{a_{n-1}\}, 1) + \\ &\qquad \mathfrak{C}(k, B, a_n, a_{n-1}, 1) = \mathfrak{C}(k, \{B\} \sqcup\!\sqcup \{a_n, a_{n-1}\}, 1) = \mathfrak{C}(1, k, \{B\} \sqcup\!\sqcup \{a_n, a_{n-1}\}) \ .\end{aligned} \tag{2.127}$$

Therefore we obtain the KK relations

$$\mathfrak{C}(1, A, k, B) = (-1)^{|A|} \sum_{\sqcup\!\sqcup} \mathfrak{C}(1, k, \{B\} \sqcup\!\sqcup \{A^T\}) \ , \tag{2.128}$$

where $|A|$ denotes the number of elements in A and A^T is the reverse ordering of A. This proofs KK relations between YM partial amplitudes. It should be mentioned that proving KK relations outside the CHY frame is non-trivial [65], confirming the advantages of this formalism.

Proving BCJ relations is also remarkably simple. Any scattering equation (2.111) can be rewritten using the insertion operator in the following way

$$\frac{f_k}{2} = - \sum_{\substack{a=1 \\ a \neq k, k-1}}^{n-1} p_k \cdot X_a\, S(a, k, a+1) - p_k \cdot X_{k-1}\, S(k-1, k, k+1)\, \theta(n-k) \ , \tag{2.129}$$

with the region momentum $X_a = (\sum_{b=1}^{a} p_b)$ and $\theta(n-k)$ is the Heavyside function, i.e. the last term is only non-zero while $k < n$. We can easily see that this formula is true by expanding all insertion operators and collecting terms, i.e. choose $k < n$

$$
\begin{aligned}
&-\sum_{\substack{a=1\\a\neq k,k-1}}^{n-1} p_k\cdot X_a\, S(a,k,a+1) - p_k\cdot X_{k-1}\, S(k-1,k,k+1) = \\
&-\sum_{\substack{a=1\\a\neq k,k-1}}^{n-1} p_k\cdot X_a \left(\frac{1}{\sigma_{ak}} + \frac{1}{\sigma_{ka+1}}\right) - p_k\cdot X_{k-1}\left(\frac{1}{\sigma_{k-1k}} + \frac{1}{\sigma_{kk+1}}\right) = \\
&-\sum_{\substack{a=1\\a\neq k}}^{n-1} p_k\cdot X_a \frac{1}{\sigma_{ak}} - \sum_{\substack{a=1\\a\neq k-1}}^{n-1} p_k\cdot X_a \frac{1}{\sigma_{ka+1}} = -\sum_{\substack{a=1\\a\neq k}}^{n-1} p_k\cdot X_a \frac{1}{\sigma_{ak}} + \sum_{\substack{a=2\\a\neq k}}^{n} p_k\cdot X_{a-1}\frac{1}{\sigma_{ak}} = \\
&-\sum_{\substack{a=2\\a\neq k}}^{n-1} p_k\cdot (X_a - X_{a-1})\frac{1}{\sigma_{ak}} - \frac{p_k\cdot X_1}{\sigma_{1k}} + \frac{p_k\cdot X_{n-1}}{\sigma_{nk}} = \sum_{\substack{a=1\\a\neq k}}^{n} \frac{p_k\cdot p_a}{\sigma_{ka}} = \frac{f_k}{2}\,,
\end{aligned}
\tag{2.130}
$$

where we used $p_k \cdot X_k = p_k \cdot X_{k-1}$ due to $p_k^2 = 0$ in the third line and $X_a - X_{a-1} = p_a$ as well as $X_{n-1} = p_n$, i.e. (2.129) is only valid on the support of momentum conservation and on-shell conditions. Now consider the expression

$$f_k\, \mathfrak{C}(1, ...k-1, k+1, ..., n), \tag{2.131}$$

where the index set inside the PT function contains all particle labels except k. With the help of (2.129) we obtain

$$\sum_{a=1}^{n-1} p_k\cdot X_a\, \mathfrak{C}(1,...,a,k,a+1,...,n) = \sum_{\sqcup} p_k\cdot X_k\, \mathfrak{C}(1,\{2,...,n-1\}\sqcup\{k\},n) \tag{2.132}$$

which if inserted into (2.120) yields zero due to the enforcing of the scattering equations therefore reproducing BCJ relations. Again, alternative proofs of BCJ relations are more involved [66].

Chapter 3

From adjacent collinear gluons to gravitons

This chapter is based on the published paper "Collinear limits beyond the leading order from the scattering equations" [109], written in collaboration with Prof. Jan Plefka and Dr. Dhritiman Nandan.

In this chapter we want to dive deeper into the connection between collinear gluons and gravitons which we have encountered in sec. 2.7. Our goal is twofold. On the one hand, we want to derive ST-one (2.103) from a field theoretic point of view, contrasting the original derivation from string theory, in order to get a deeper insight into the inner workings of ST-three. On the other hand, we want to narrow the gap between single soft- and collinear gluon limits in terms of established knowledge, which is highly rich for single soft gluon limits but very deserted regarding collinear gluon limits for which only the leading, singular behaviors are generally known. Of special importance in this regard are questions of universality and factorization, i.e. whether a sub-leading collinear theorem exists.

We will start by computing the sub-leading collinear gluon limit for a tree level pure YM partial amplitude. This is very hard from a Feynman-diagrammatic point of view due to the size of the involved analytic expressions. We will circumvent this issue by using the CHY formalism 2.8 which allows for an analytic treatment of a generic scattering amplitude in arbitrary dimensions in a compact way. Furthermore, it also allows for an easy study of universality due to the possibility to interchange particle types via simple replacements of CHY building blocks. However, the CHY formalism introduces new complications. By mapping external kinematic data onto punctures on the Riemann sphere, it becomes necessary to analyze and establish new techniques on how to the collinear limit, as a special kinematic configuration, affects the positions of the associated punctures.

Afterwards, we will tackle a generic proof of ST-one by deriving an alternative linear combination which mimics its the analytic forms. This will allow us to interpret ST-one as a gauge invariance condition for EYM amplitudes while also constraining a sub-leading collinear theorem.

3.1 General considerations

We begin by briefly discussing how adjacent collinear gluon limits affect tree level scattering amplitudes, while also making first introductions to our notation for the upcoming analysis. Out goal is to establish a generic insight on why connections between collinear gluons and gravitons should exist. Collinear gluon kinematics are schematically of the form (let us say the adjacent legs i and $i+1$ go collinear)

$$p_i = z\,p + \delta p_i \quad , \quad p_{i+1} = (1-z)\,p + \delta p_{i+1} \ , \tag{3.1}$$

where we have introduced the momentum fraction $0 \le z \le 1$, the massless fused momentum p and some small perturbation δp which keeps track of the divergence in the scattering amplitude. Equivalently, the associated polarizations also undergo the formal collinear expansion

$$\varepsilon_i = \varepsilon_p + \delta\varepsilon_i \quad , \quad \varepsilon_{i+1} = \varepsilon_p + \delta\varepsilon_{i+1} \ , \tag{3.2}$$

where we call ε_p the fused polarization. A partial tree level scattering amplitude will then have a perturbative expansion of the form

$$\begin{aligned} \mathrm{A}_n^{p_i||p_{i+1}}(1,2,...,i,i+1,...,n) &= \varepsilon_i^\mu\,\varepsilon_{i+1}^\nu \mathrm{A}_{n,\mu\nu}^{p_i||p_{i+1}}(1,2,...,i,i+1,...,n) \\ &= (\varepsilon_p^\mu + \delta\varepsilon_i^\mu)(\varepsilon_p^\nu + \delta\varepsilon_{i+1}^\nu)\left(\mathrm{A}_{n,\mu\nu}^{(0)} + \mathrm{A}_{n,\mu\nu}^{(1)} + ...\right) \ , \end{aligned} \tag{3.3}$$

where we have extracted the collinear polarization vectors from the amplitude. The divergence will be encoded in the object $\mathrm{A}_n^{(0)} = \varepsilon_p^\mu\,\varepsilon_p^\nu\,\mathrm{A}_{n,\mu\nu}^{(0)}$. The full sub leading part is given by

$$\mathrm{A}_n^{(1)} = \varepsilon_p^\mu\,\varepsilon_p^\nu\,\mathrm{A}_{n,\mu\nu}^{(1)} + \left(\varepsilon_p^\mu\,\delta\varepsilon_{i+1}^\nu + \delta\varepsilon_i^\mu\,\varepsilon_p^\nu\right)\mathrm{A}_{n,\mu\nu}^{(0)} \ . \tag{3.4}$$

We can directly see that it exists an explicit connection to gravitons since the above expression involves a graviton polarization tensor $\varepsilon_p^{\mu\nu} = \varepsilon_p^\mu\,\varepsilon_p^\nu$. Therefore, we can easily imagine that formulae like ST-one (2.103) might exist. Note that we have not specified any theory for the above formulae. This is a hint for universality of ST-one.

3.2 Collinear limits on the Riemann sphere

In this section we will define and establish the technical details for the upcoming computation of an adjacent collinear gluon limit in the CHY frame.

3.2.1 Strategy for the approach in the CHY frame

Our strategy is inspired by the study of double soft gluon limits in CHY [96]. Collinear gluons can be adjacent or non-adjacent. For both cases we apply the same steps of computation. Assume that two momenta p_i and p_j are collinear, i.e. $p_i + p_j = p$ with $p^2 = 0$. Their associated punctures on the Riemann sphere, i.e. σ_i and σ_j, are then subject to the change of variables

$$\sigma_1 = \sigma_p - \frac{\xi}{2} \quad , \quad \sigma_2 = \sigma_p + \frac{\xi}{2} \ , \tag{3.5}$$

together with a change of the associated scattering equations in the CHY measure

$$\delta(f_i)\,\delta(f_j) = \delta(f_-)\,\delta(f_+)\ . \tag{3.6}$$

As a result the CHY amplitude reads

$$\begin{aligned} \mathrm{A}_n^{\mathrm{Th}}(\{p,\varepsilon\}) &= \int_{\mathbb{C}\mathrm{P}^1} d\mu_{n-2}\, d\sigma_i\, d\sigma_j\, \delta(f_i)\,\delta(f_j)\, \mathcal{I}_n^{\mathrm{Th}}(\{p,\varepsilon,\sigma\}) \\ &= 2\int_{\mathbb{C}\mathrm{P}^1} d\mu_{n-2}\, d\xi\, d\sigma_p\, \delta(f_-)\,\delta(f_+)\, \mathcal{I}_n^{\mathrm{Th}}(\{p,\varepsilon,\sigma\})\ . \end{aligned} \tag{3.7}$$

As a next step we perform the ξ integral thus localizing the integral on the solutions ξ_{sol} of $f_-\big|_{\xi_{\mathrm{sol}}} = 0$, i.e.

$$\mathrm{A}_n^{\mathrm{Th}}(\{p,\varepsilon\}) = 2\sum_{\xi_{\mathrm{sol}}} \int d\mu_{n-2}\, d\sigma_p\, \delta(f_+)\, \mathcal{J}\, \mathcal{I}_n^{\mathrm{Th}}\ , \tag{3.8}$$

where we used

$$\delta(f_-) = \sum_{\xi_{\mathrm{sol}}} \frac{\delta(\xi-\xi_{\mathrm{sol}})}{\left|\frac{\partial f_-}{\partial \xi}\right|_{\xi=\xi_{\mathrm{sol}}}} = \sum_{\xi_{\mathrm{sol}}} \delta(\xi-\xi_{\mathrm{sol}})\, \mathcal{J}\ , \tag{3.9}$$

such that $\mathcal{J}$ is the Jacobian of the variable transformation defined above. Using such a strategy seems opaque at this point but will prove successful as we shall see in the upcoming sections.

3.2.2 Collinear Kinematics

As we are aiming at the sub-leading adjacent collinear gluon limit, we want to take control of the divergent structure of the leading contribution via an appropriate definition of collinear kinematics. As we also aim at an understanding of the formulae presented by S. Stieberger and T. Taylor [38], i.e. ST-one (2.103), we shall use similar kinematic definitions. All kinematic definitions in [38] are presented using spinor-helicity variables, which are key components in the spinor-helicity formalism [1]. Its purpose is a unified description of massless on-shell momenta and polarization vectors in 4d Minkowski spacetime through the notion of two different types of spinor brackets. We will focus on its most practical results. In a nutshell, momenta are written as

$$p_i = |i\rangle[i|\ , \tag{3.10}$$

with products

$$2\,p_i\cdot p_j = \langle j\,i\rangle[i\,j]\ . \tag{3.11}$$

The spinor brackets are antisymmetric, i.e. $\langle j\,i\rangle = -\langle i\,j\rangle$, $[i\,j] = -[j\,i]$ which implies $\langle j\,j\rangle = [j\,j] = 0$. Polarizations are written as

$$\varepsilon_a^+ = +\sqrt{2}\,\frac{|a]\,\langle r_a|}{\langle r_a\,a\rangle} \quad , \quad \varepsilon_a^- = -\sqrt{2}\,\frac{|a\rangle\,[r_a|}{[r_a\,a]}\quad , \tag{3.12}$$

[1] For a textbook treatment see [57–60]

where the reference spinors $|r_a], |r_a\rangle$ can be chosen freely as long they are not proportional to $|p_a], |p_a\rangle$. The products are then

$$\begin{aligned} &\varepsilon_a^+ \cdot \varepsilon_b^- = -2\,\frac{[r_b\,a]\,\langle r_a\,b\rangle}{\langle r_a\,a\rangle\,[r_b\,b]} \quad , \quad \varepsilon_a^- \cdot \varepsilon_b^- = 2\,\frac{[r_a\,r_b]\,\langle b\,a\rangle}{\langle r_a\,a\rangle\,[r_b\,b]} \quad , \quad \varepsilon_a^+ \cdot \varepsilon_b^+ = 2\,\frac{[a\,b]\,\langle r_b\,r_a\rangle}{\langle r_a\,a\rangle\,[r_b\,b]} \\ &\varepsilon_a^+ \cdot p_b = \sqrt{2}\,\frac{\langle b\,r_a\rangle\,[a\,b]}{\langle r_a\,a\rangle} \quad , \quad \varepsilon_a^- \cdot p_b = -\sqrt{2}\,\frac{\langle a\,b\rangle\,[b\,r_a]}{[r_a\,a]} \ , \end{aligned} \tag{3.13}$$

which satisfy all desired on-shell conditions (2.8).

Without loss of generality, we choose the particles with momenta p_1 and p_2 collinear. Their spinors are defined as [38]

$$\begin{aligned} |1\rangle &= \mathbf{c}\,|p\rangle - \epsilon\,\mathbf{s}\,|r\rangle \quad , \quad |1] = \mathbf{c}\,|p] - \epsilon\,\mathbf{s}\,|r] \quad , \quad \mathbf{c} = \cos\phi \ , \\ |2\rangle &= \mathbf{s}\,|p\rangle + \epsilon\,\mathbf{c}\,|r\rangle \quad , \quad |2] = \mathbf{s}\,|p] + \epsilon\,\mathbf{c}\,|r] \quad , \quad \mathbf{s} = \sin\phi \ , \end{aligned} \tag{3.14}$$

where we use the perturbation parameter ϵ to probe the sub-leading collinear effect as $\epsilon \to 0$ and we introduced the parameters $\mathbf{c} = \cos\,\phi$ and $\mathbf{s} = \sin\,\phi$ into the role of momentum fraction with arbitrary phase ϕ. Additionally, we include a reference spinor with index r which is not proportional to p. This ansatz translates into the covariant momenta

$$\begin{aligned} p_1 &= \mathbf{c}^2\,p - \epsilon\,\mathbf{cs}\,q + \epsilon^2\,\mathbf{s}^2\,r \ , \\ p_2 &= \mathbf{s}^2\,p + \epsilon\,\mathbf{cs}\,q + \epsilon^2\,\mathbf{c}^2\,r \ , \end{aligned} \tag{3.15}$$

where we defined $q = |p\rangle[r| + |r\rangle[p|$. The fused momentum p now obtains corrections in ϵ^2. The reference spinor r translates into a light-like reference momentum satisfying $p \cdot r \neq 0$. We summarize some important kinematic expressions

$$(p_1+p_2) = p+\epsilon^2\,r \quad , \quad (p_1-p_2) = (\mathbf{c}^2-\mathbf{s}^2)(p-\epsilon^2\,r)-2\epsilon\,\mathbf{cs}\,q \quad , \quad (p_1\cdot p_2) = \epsilon^2\,p\cdot r\,. \tag{3.16}$$

Collinear momenta translate into collinear polarizations. We choose the same reference spinor r for all polarizations thus obtaining

$$\varepsilon_a^\pm \cdot r = 0 \quad , \quad \forall a\ . \tag{3.17}$$

Additionally, we expand the collinear polarizations

$$\varepsilon_1^\pm = \varepsilon_p^\pm - \epsilon\,\frac{\mathbf{s}}{\mathbf{c}}\,\tilde{\varepsilon}_{p,r}^\pm \quad , \quad \varepsilon_2^\pm = \varepsilon_p^\pm + \epsilon\,\frac{\mathbf{c}}{\mathbf{s}}\,\tilde{\varepsilon}_{p,r}^\pm \ , \tag{3.18}$$

where we define the fused polarization $\varepsilon_p^\pm$ and

$$\tilde{\varepsilon}_{p,r}^\pm := \begin{cases} +\sqrt{2}\,\frac{|r\rangle\,[r|}{\langle r\,p\rangle} & \text{pos. helicity} \\ -\sqrt{2}\,\frac{|r\rangle\,[r|}{[r\,p]} & \text{neg. helicity}\ . \end{cases} \tag{3.19}$$

The identity

$$\tilde{\varepsilon}_{p,r}^{h_p} \cdot p = -\varepsilon_p^{h_p} \cdot q \ , \tag{3.20}$$

follows directly from $\varepsilon_1^{h_1} \cdot p_1 = \varepsilon_2^{h_2} \cdot p_2 = 0$ and it will play an important role in later sections.

3.2.3 Scattering Equations with Collinear Gluons

Having established the precise definition of collinear kinematics, we can continue with an analysis on how the scattering equations in the CHY measure (2.110) are affected. We stress that gauge fixing modular invariance cannot involve the collinear punctures since one would exclude deformations of them. This is intuitively inconsistent with the CHY philosophy linking changing kinematic behaviors to changes of puncture positions on the Riemann sphere.

As already discussed in the beginning of this chapter we perform the variable change

$$\sigma_1 = \sigma_p - \frac{\xi}{2} \quad , \quad \sigma_2 = \sigma_p + \frac{\xi}{2} \, , \tag{3.21}$$

while also changing the scattering equations f_1 and f_2 such that the full measure reads

$$\begin{aligned} d\mu_n &= d\mu_{n-2}\, d\Omega \\ d\Omega &= d\sigma_1\, d\sigma_2\, \delta(f_1)\, \delta(f_2) = 2\, d\sigma_p\, d\xi\, \delta(f_+)\, \delta(f_-) \, , \end{aligned} \tag{3.22}$$

where we now provide an explicit choice of transformation, i.e.

$$f_+ = (f_1 + f_2) \quad , \quad f_- = (f_1 - f_2) - (\mathbf{c}^2 - \mathbf{s}^2)(f_1 + f_2) \ . \tag{3.23}$$

This particular choice of f_- will unveil its advantage at a later point in this section. The relevant scattering equations now read

$$\begin{aligned} f_a &= \sum_{\substack{b=3 \\ b\neq a}}^{n} \frac{2\, p_a \cdot p_b}{\sigma_a - \sigma_b} + \frac{2\, p_a \cdot p_1}{\sigma_a - \sigma_p + \frac{\xi}{2}} + \frac{2\, p_a \cdot p_2}{\sigma_a - \sigma_p - \frac{\xi}{2}} \, , \qquad a \neq 1, 2 \\ f_1 - f_2 &= \sum_{b=3}^{n} \Big(\frac{2\, p_b \cdot p_1}{\sigma_p - \frac{\xi}{2} - \sigma_b} - \frac{2\, p_b \cdot p_2}{\sigma_p + \frac{\xi}{2} - \sigma_b} \Big) - 4\, \frac{p_1 \cdot p_2}{\xi} \\ f_1 + f_2 &= \sum_{b=3}^{n} \Big(\frac{2\, p_b \cdot p_1}{\sigma_p - \frac{\xi}{2} - \sigma_b} + \frac{2\, p_b \cdot p_2}{\sigma_p + \frac{\xi}{2} - \sigma_b} \Big) \, . \end{aligned} \tag{3.24}$$

Next, we want to determine solutions ξ_{sol} to f_-. This is a highly non-trivial task due to the non-linear nature of the scattering equations. We approach this issue by performing a numerical analysis [2] for some non-trivial multiplicity scenarios, i.e. $n = 5, 6, 7$ and 8 particles. We find that the set of solutions ξ_{sol} splits into two categories. On the one hand, we have $2(n-4)!$ degenerate solutions with vanishing $\xi_{\text{deg}} = \sigma_{21} = \mathcal{O}(\epsilon)$. On the other hand, we obtain $(n-5)(n-4)!$ non-degenerate solutions with finite $\xi_{\text{non-deg}} = \mathcal{O}(1)$. Additionally, we find that the non-degenerate solutions are irrelevant for the computation of the sub-leading collinear limit since their contribution to the full amplitude is of sub-sub-leading order. Thus, it is sufficient to focus on the degenerate solutions and we make the ansatz

$$\xi_{\text{deg}} = \epsilon\, \xi_1 + \epsilon^2\, \xi_2 + \mathcal{O}(\epsilon^3) \ , \tag{3.25}$$

[2] Performance is accelerated by using the polynomial form of the scattering equations presented in [110]

i.e. $\sigma_1 \to \sigma_2$[3]. With this ansatz, the scattering equations read

$$\begin{aligned}
f_a &= \bar{f}_a + \epsilon\left[-(\mathbf{c}^2-\mathbf{s}^2)\frac{\xi_1}{2}\mathcal{P}_{2,a}\right] + \mathcal{O}(\epsilon^2)\,, \qquad a \neq 1,2 \\
f_+ &= \bar{f}_p + \epsilon\left[(\mathbf{c}^2-\mathbf{s}^2)\frac{\xi_1}{2}\mathcal{P}_2\right] + \mathcal{O}(\epsilon^2) \\
f_- &= \epsilon\left[2\mathbf{c}^2\mathbf{s}^2\xi_1\mathcal{P}_2 - 2\mathbf{cs}\mathcal{Q}_1 - \frac{4\,(p\cdot r)}{\xi_1}\right] \\
&\quad + \epsilon^2\left[4\,(p\cdot r)\frac{\xi_2}{\xi_1^2} - 2(\mathbf{c}^2-\mathbf{s}^2)\mathcal{R}_1 + 2\mathbf{c}^2\mathbf{s}^2\xi_2\mathcal{P}_2 + \xi_1(\mathbf{c}^2-\mathbf{s}^2)\mathbf{cs}\mathcal{Q}_2\right] + \mathcal{O}(\epsilon^3)\,,
\end{aligned} \tag{3.26}$$

where we have introduced the modified scattering equations for $n-1$ particles as

$$\bar{f}_a = \sum_{\substack{b=0\\ b\neq a}}^{n} \frac{2\,p_a\cdot p_b}{\sigma_a-\sigma_b} \quad , \quad \bar{f}_p = \sum_{b=3}^{n}\frac{2\,p\cdot p_b}{\sigma_p-\sigma_b} \quad , \quad 3\leq a\leq n\,, \tag{3.27}$$

along with the short-hand notations,

$$\begin{aligned}
\mathcal{P}_{i,a} &= \frac{2\,p\cdot p_a}{(\sigma_p-\sigma_a)^i} \quad , \quad \mathcal{P}_i = \sum_{b=3}^{n}\frac{2\,p\cdot p_b}{(\sigma_p-\sigma_b)^i} \quad , \quad \mathcal{R}_i = \sum_{b=3}^{n}\frac{2\,r\cdot p_b}{(\sigma_p-\sigma_b)^i} \\
\mathcal{R}_{i,a} &= \frac{2\,r\cdot p_a}{(\sigma_p-\sigma_b)^i} \quad , \quad \mathcal{Q}_i = \sum_{b=3}^{n}\frac{2\,q\cdot p_b}{(\sigma_p-\sigma_b)^i} \quad , \quad \mathcal{Q}_{i,a} = \frac{2\,q\cdot p_a}{(\sigma_p-\sigma_b)^i} \quad , \quad i\geq 1\,,
\end{aligned} \tag{3.28}$$

for compactness. For $\bar{f}_a$ the sum starts from 0 where $p_0 = p$ and thus the sum runs over the set of momenta is $\{p, p_3, p_4, \ldots, p_n\}$. Note that $\mathcal{P}_1 = \bar{f}_p = \mathcal{O}(\epsilon)$ and that f_- takes the form

$$f_- = \epsilon a_1(\xi_1) + \epsilon a_2(\xi_1,\xi_2) + \ldots = 0\,, \quad \to \quad a_i(\xi_1, ..., \xi_i) = 0\,, \tag{3.29}$$

entirely fixing the solutions for ξ_i recursively and in fact fixing all ξ_i , $i \geq 2$ to be functions of ξ_1 only. More explicitly, we can solve for ξ_1 from

$$a_1 = -2\,(p\cdot r) - \mathbf{cs}\mathcal{Q}_1\xi_1 + \mathbf{c}^2\mathbf{s}^2\xi_1^2\mathcal{P}_2 = 0\,, \tag{3.30}$$

with two solutions

$$\begin{aligned}
&\xi_1 = \xi_1^{\pm} = x_1 \pm \sqrt{x_2} \quad , \\
&x_1 = \frac{\mathcal{Q}_1}{2\mathbf{cs}\mathcal{P}_2} \quad , \quad x_2 = \frac{\mathcal{Q}_1^2 + 8(p\cdot r)\mathcal{P}_2}{4(\mathbf{c}^2\mathbf{s}^2)\mathcal{P}_2^2}\,.
\end{aligned} \tag{3.31}$$

From $a_2 = 0$ we get

$$\frac{\xi_2}{\xi_1^2} = \frac{\mathbf{c}^2-\mathbf{s}^2}{2}\,\frac{2\mathcal{R}_1 - \mathbf{cs}\,\xi_1\,\mathcal{Q}_2}{2\,p\cdot r + \mathbf{c}^2\mathbf{s}^2\,\xi_1^2\,\mathcal{P}_2}\,. \tag{3.32}$$

We can see explicitly that ξ_2 is entirely fixed by ξ_1. This recursive property explains our specific choice of f_-.

[3] It was actually known before that the degenerate solutions imply collinearity [110] but it was unknown whether the opposite is true.

3.3 Collinear expansion of CHY YM building blocks

In this section we will make our final preparations to compute the adjacent gluon sub-leading collinear limit in the CHY frame. We already established that the amplitude takes the analytic form (3.8). For pure YM amplitudes we have

$$\mathrm{A}_n^{\mathrm{YM}}(1,2,...,n) = 2\sum_{\xi_1}\int d\mu_{n-2}\, d\sigma_p\, \delta(f_+)\, \mathcal{J}\, \mathfrak{C}_n(1,2,...,n)\ \mathrm{Pf}'\,\Psi_n\ . \tag{3.33}$$

Note that the sum over degenerate solutions has reduced to the sum of solutions to ξ_1 which is a consequence of the observation that the solutions of ξ_1 fix all ξ_i.

We now expand the building blocks in ϵ while inserting all expansions from the last two sections.

3.3.1 Jacobian

We write

$$\mathcal{J} = \mathcal{J}_0 + \epsilon\,\mathcal{J}_1 + \mathcal{O}(\epsilon^2)\ , \tag{3.34}$$

with

$$\begin{aligned}\mathcal{J}_0 &= \frac{1}{2}\,\frac{\xi_1^2}{2(p\cdot r) + \mathbf{c}^2\mathbf{s}^2\mathcal{P}_2\,\xi_1^2}\ ,\\ \mathcal{J}_1 &= \mathcal{J}_0^2\left(8(p\cdot r)\frac{\xi_2}{\xi_1^3} - \mathbf{cs}(\mathbf{c}^2-\mathbf{s}^2)\mathcal{Q}_2\right)\ .\end{aligned} \tag{3.35}$$

Note that we can also write (3.32) as

$$\xi_2 = (\mathbf{c}^2-\mathbf{s}^2)\mathcal{J}_0\,(2\mathcal{R}_1 - \mathbf{cs}\,\xi_1\,\mathcal{Q}_2)\ . \tag{3.36}$$

3.3.2 Measure.

The CHY measure takes the form

$$d\mu_{n-2}\, d\sigma_p\, \delta(f_+) = d\mu_{n-1}^{(0)} + \epsilon\, d\mu_{n-1}^{(1)} + \mathcal{O}(\epsilon^2)\ , \tag{3.37}$$

with

$$d\mu_{n-1}^{(0)} = d\mu_{n-2}\, d\sigma_p\, \delta(\bar{f}_p)\ , \tag{3.38}$$

and

$$d\mu_{n-1}^{(1)} = (\mathbf{c}^2-\mathbf{s}^2)\, d'\sigma_{n-2}\, d\sigma_p\, \frac{\xi_1}{2}\left(\mathcal{P}_2\,\delta'(\bar{f}_p)\Delta_{n-2}^{\prime(0)} - \delta(\bar{f}_p)\sum_{a=3}^{n}{}' \mathcal{P}_{2,a}\delta'(\bar{f}_a)\,\Delta_{n-3,a}^{\prime(0)}\right)\ , \tag{3.39}$$

where

$$\delta'(x) = \frac{\partial\,\delta(x)}{\partial\, x}\quad ,\quad \Delta_{n-3,a}^{\prime(0)} = \prod_{\substack{i=3\\ i\neq a}}^{n}\delta(\bar{f}_a)\quad ,\quad \sum_{a=3}^{n}{}' = (\sigma_{ij}\,\sigma_{jk}\,\sigma_{ki})\sum_{\substack{a=3\\ a\neq i,j,k}}^{n}\ . \tag{3.40}$$

The definition of the primed sum results from gauge fixing modular invariance.

3.3.3 Parke-Taylor factor.

We obtain,

$$\begin{aligned}
&\mathfrak{C}_n(1,2,...,n) = \frac{1}{\epsilon}\mathfrak{C}_n^{(0)} + \mathfrak{C}_n^{(1)} + \mathcal{O}(\epsilon) \ , \\
&\mathfrak{C}_n^{(0)} = -\frac{1}{\xi_1}\mathfrak{C}_{n-1}(p,3,...,n) \quad , \quad \mathfrak{C}_n^{(1)} = \mathfrak{C}_{n-1}(p,3,...,n)\left(\frac{\xi_2}{\xi_1^2} + \frac{1}{2}\,S(n,p,3)\right) , \\
&\mathfrak{C}_{n-1}(p,3,...,n) = \frac{1}{\sigma_{34}\ldots\sigma_{np}\,\sigma_{p3}} \quad ,
\end{aligned} \tag{3.41}$$

where $S(n,p,3)$ is the insertion operator (2.121).

3.3.4 The matrix Ψ_n.

We write the matrix (2.114) with more emphasis on the collinear entries

$$\Psi_n = \begin{pmatrix}
0 & A_{12} & A_{1b} & -C_{11}^{h_1} & -C_{21}^{h_2} & -C_{d1}^{h_d} \\
A_{21} & 0 & A_{2b} & -C_{12}^{h_1} & -C_{22}^{h_2} & -C_{d2}^{h_d} \\
A_{a1} & A_{a2} & A_{ab} & -C_{1a}^{h_1} & -C_{2a}^{h_2} & -C_{da}^{h_d} \\
C_{11}^{h_1} & C_{12}^{h_1} & C_{1b}^{h_1} & 0 & B_{12}^{h_1|h_2} & B_{1d}^{h_1|h_d} \\
C_{21}^{h_2} & C_{22}^{h_2} & C_{2b}^{h_2} & B_{21}^{h_2|h_1} & 0 & B_{2d}^{h_2|h_d} \\
C_{c1}^{h_c} & C_{c2}^{h_c} & C_{cb}^{h_c} & B_{c1}^{h_c|h_1} & B_{c2}^{h_c|h_2} & B_{cd}^{h_c|h_d}
\end{pmatrix} . \tag{3.42}$$

We expand the components individually:

1. ***A*-matrix:**

$$\begin{aligned}
&A_{1b} = \mathbf{c}^2\,A_{pb} + \epsilon\left(\frac{\xi_1}{2}\,\mathbf{c}^2\,A_{pb}^{(2)} - c\,s\,Q_b\right) \ , \ A_{2b} = \mathbf{s}^2\,A_{pb} - \epsilon\left(\frac{\xi_1}{2}\,\mathbf{s}^2\,A_{pb}^{(2)} - c\,s\,Q_b\right) \ , \\
&A_{12} = -\epsilon\frac{2\,p\cdot r}{\xi_1} + \epsilon^2\,2\,p\cdot r\frac{\xi_2}{\xi_1^2} \ .
\end{aligned} \tag{3.43}$$

2. ***B*-matrix:**

$$\begin{aligned}
&B_{1b}^{h_1|h_b} = B_{pb}^{h_1|h_b} + \epsilon\,\frac{\xi_1}{2}\,B_{pb}^{h_1|h_b,(2)} \quad , \quad B_{2b}^{h_2|h_b} = B_{pb}^{h_2|h_b} - \epsilon\,\frac{\xi_1}{2}\,B_{pb}^{h_2|h_b,(2)} \ , \\
&B_{12}^{h_1|h_2} = \begin{cases} 0 & h_1 \ = \ h_2 \\ \frac{2}{\epsilon\,\xi_1} - \frac{2\,\xi_2}{\xi_1^2} & h_1 \ \neq \ h_2 \end{cases} .
\end{aligned} \tag{3.44}$$

3. **$\boldsymbol{C}$-matrix:**

$$
\begin{aligned}
C_{b1}^{h_b} &= \mathbf{c}^2\, C_{bp}^{h_b} - \epsilon \left(\frac{\xi_1}{2}\, \mathbf{c}^2\, C_{bp}^{h_b,(2)} + \mathbf{cs}\, C_{bq}^{h_b} \right) \ , \ C_{b2}^{h_b} = \mathbf{s}^2\, C_{bp}^{h_b} - \epsilon \left(\frac{\xi_1}{2}\, \mathbf{s}^2\, C_{bp}^{h_b,(2)} - \mathbf{cs}\, C_{bq}^{h_b} \right) \ , \\
C_{1b}^{h_1} &= C_{pb}^{h_1} + \epsilon \left(\frac{\xi_1}{2} C_{pb}^{h_1,(2)} - \frac{\mathbf{s}}{\mathbf{c}}\, E_b^{h_1} \right) \quad , \quad C_{2b}^{h_2} = C_{pb}^{h_2} + \epsilon \left(- \frac{\xi_1}{2} C_{pb}^{h_2,(2)} + \frac{\mathbf{c}}{\mathbf{s}}\, E_b^{h_2} \right) \ , \\
C_{12}^{h_1} &= \frac{\mathbf{s}}{\mathbf{c}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_1} \cdot p}{\xi_1} - \epsilon\, \frac{\mathbf{s}}{\mathbf{c}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_1} \cdot p}{\xi_1}\, \frac{\xi_2}{\xi_1} \quad , \quad C_{21}^{h_2} = \frac{\mathbf{c}}{\mathbf{s}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_2} \cdot p}{\xi_1} - \epsilon\, \frac{\mathbf{c}}{\mathbf{s}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_2} \cdot p}{\xi_1}\, \frac{\xi_2}{\xi_1} \ , \\
C_{11}^{h_1} &= C_{pp}^{h_1} - \frac{\mathbf{s}}{\mathbf{c}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_1} \cdot p}{\xi_1} + \epsilon \left(\frac{\xi_1}{2}\, C_{pp}^{h_1,(2)} + \frac{\mathbf{s}}{\mathbf{c}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_1} \cdot p}{\xi_1}\, \frac{\xi_2}{\xi_1} + \frac{\mathbf{s}}{\mathbf{c}}\, E^{h_1} \right) \ , \\
C_{22}^{h_2} &= C_{pp}^{h_2} - \frac{\mathbf{c}}{\mathbf{s}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_2} \cdot p}{\xi_1} + \epsilon \left(- \frac{\xi_1}{2}\, C_{pp}^{h_2,(2)} + \frac{\mathbf{c}}{\mathbf{s}}\, \frac{2\, \tilde{\varepsilon}_{p,r}^{h_2} \cdot p}{\xi_1}\, \frac{\xi_2}{\xi_1} - \frac{\mathbf{c}}{\mathbf{s}}\, E^{h_2} \right) \ , \\
C_{aa}^{h_a} &= C_{aa}^{h_a} + \epsilon\, (\mathbf{c}^2 - \mathbf{s}^2)\, \frac{\xi_1}{2}\, C_{ap}^{h_a,(2)} \ ,
\end{aligned}
\tag{3.45}
$$

where we also introduced the short-hand notation

$$
\begin{aligned}
C_{bq}^{h_b} &= \frac{2\, \varepsilon_b^{h_b} \cdot q}{\sigma_b - \sigma_p} \quad , \quad A_{pb}^{(i)} = \frac{2\, p \cdot p_b}{(\sigma_p - \sigma_b)^i} \quad , \quad B_{pb}^{h_p|h_b,(i)} = \frac{2\, \varepsilon_p^{h_p} \cdot \varepsilon_b^{h_b}}{(\sigma_p - \sigma_b)^i} \ , \\
C_{bp}^{h_b,(i)} &= \frac{2\, \varepsilon_b^{h_b} \cdot p}{(\sigma_b - \sigma_p)^i} \quad , \quad C_{pb}^{h_p,(i)} = \frac{2\, \varepsilon_p^{h_p} \cdot p_b}{(\sigma_p - \sigma_b)^i} \quad , \quad C_{pp}^{h_p,(i)} = - \sum_{b=3}^{n} \frac{2\, \varepsilon_p^{h_p} \cdot p_b}{(\sigma_p - \sigma_b)^i} \ , \\
E_b^{h_r} &= \frac{2\, \tilde{\varepsilon}_{p,r}^{h_r} \cdot p_b}{\sigma_{pb}} \quad , \quad E^{h_r} = \sum_{b=3}^{n} E_b^{h_r} \ .
\end{aligned}
\tag{3.46}
$$

3.3.5 The Pfaffian $\mathrm{Pf}'\,\Psi_n$.

Knowing the expansion of the matrix entries, the expansion of the Pfaffian follows directly from its recursive definition

$$
\mathrm{Pf}'\,\Psi_n = (-1)^{n+1} \sum_{\substack{j=1 \\ j\neq i}}^{2n} (-1)^{i+j+1+\theta(j-i)}\, \Psi_{ij}\ \mathrm{Pf}'\,\Psi_n^{i,j} \ . \tag{3.47}
$$

In this formula we expand along row j, $\Theta(j-i)$ is the Heaviside function, Ψ_{ij} is the matrix element at position (i,j) and $\Psi_n^{i,j}$ is the matrix Ψ_n with both deleted rows and columns i,j. By definition $\mathrm{Pf}(\Psi_n^{1,\ldots,n}) = 1$. For the sub-leading order we will also need the definition of the derivative

$$
\frac{\partial}{\partial \epsilon}\, \mathrm{Pf}'\,\Psi_n = (-1)^{n+1} \sum_{i=1}^{2n-1} \sum_{j=i}^{2n} (-1)^{i+j+1} \frac{\partial \Psi_{i,j}}{\partial \epsilon}\, \mathrm{Pf}'\,\Psi_n^{i,j} \ . \tag{3.48}
$$

For the leading order we have two distinct cases for the helicity.

Equal helicity case

In this case we have $h_1 = h_2 = h$. Our choice of reference vector r for all polarizations sets $B_{12}^{h_1|h_2} = 0$. The computations simplify dramatically if we perform the following manipulations:

1. Add row/column 1 to 2.
2. Subtract $\mathbf{c}^2$ times the new second row/column from the first row/column.

Doing so, we obtain

$$\Psi_n = \begin{pmatrix} 0 & A_{12} & \epsilon\,\tilde{A}_{1b} & -\mathbf{s}^2\,C_{11}^h + \mathbf{c}^2\,C_{12}^h & \mathbf{c}^2\,C_{22}^h - \mathbf{s}^2\,C_{21}^h & -\epsilon\,\tilde{C}_{d1}^{h_d} \\ A_{21} & 0 & \tilde{A}_{2b} & -C_{12}^h - C_{11}^h & -C_{22}^h - C_{21}^h & -\tilde{C}_{d2}^{h_d} \\ \epsilon\,\tilde{A}_{a1} & \tilde{A}_{a2} & A_{ab} & -C_{1a}^h & -C_{2a}^h & -C_{da}^{h_d} \\ \mathbf{s}^2\,C_{11}^h - \mathbf{c}^2\,C_{12}^h & C_{12}^h + C_{11}^h & C_{1b}^h & 0 & 0 & B_{1d}^{h|h_d} \\ \mathbf{s}^2\,C_{21}^h - \mathbf{c}^2 C_{22}^h & C_{22}^h + C_{21}^h & C_{2b}^h & 0 & 0 & B_{2d}^{h|h_d} \\ \epsilon\,\tilde{C}_{c1}^{h_c} & \tilde{C}_{c2}^{h_c} & C_{cb}^{h_c} & B_{c1}^{h_c|h} & B_{c2}^{h_c|h} & B_{cd}^{h_c|h_d} \end{pmatrix}, \tag{3.49}$$

with

$$\begin{aligned} \tilde{A}_{1b} &= \xi_1 \mathbf{s}^2\mathbf{c}^2 2A_{pb}^{(2)} - \mathbf{cs}\mathcal{Q}_b + \mathcal{O}(\epsilon)\ , \\ \tilde{A}_{2b} &= A_{pb} + \epsilon\,(\mathbf{c}^2 - \mathbf{s}^2)\,\frac{\xi_1}{2} A_{pb}^{(2)} + \mathcal{O}(\epsilon^2)\ , \\ \tilde{C}_{d1}^{h_d} &= \xi_1 \mathbf{s}^2\mathbf{c}^2 C_{dp}^{h_d,(2)} + \mathbf{cs} C_{dq}^{h_d} + \mathcal{O}(\epsilon)\ , \\ \tilde{C}_{d2}^{h_d} &= C_{dp}^{h_d} - \epsilon\,(\mathbf{c}^2 - \mathbf{s}^2)\,\frac{\xi_1}{2} C_{dp}^{h_d,(2)} + \mathcal{O}(\epsilon^2)\ . \end{aligned} \tag{3.50}$$

Note that A_{12} is also of $\mathcal{O}(\epsilon)$. We expand the Pfaffian along the first row/column and write

$$\mathrm{Pf}'\,\Psi_n = \mathrm{Pf}'^{(0)}\,\Psi_n + \epsilon\,\mathrm{Pf}'^{(1)}\,\Psi_n + \mathcal{O}(\epsilon^2)\ . \tag{3.51}$$

The leading contribution is thus

$$\begin{aligned} \mathrm{Pf}'^{(0)}\,\Psi_n &= \frac{-1}{\sqrt{2}}(\mathbf{s}^2\,C_{11}^h - \mathbf{c}^2\,C_{12}^h)\,\mathrm{Pf}'(\Psi_n^{1,n+1}) - \frac{1}{\sqrt{2}}\,(\mathbf{c}^2\,C_{22}^h - \mathbf{s}^2\,C_{21}^h)\,\mathrm{Pf}'(\Psi_n^{1,n+2}) \\ &= -\frac{1}{\sqrt{2}}\,(C_{pp}^h - C_{12}^h - C_{21}^h)\,\mathrm{Pf}'(\Psi_{n-1}) = -\frac{1}{\sqrt{2}}\left(C_{pp}^h - \frac{2\,\tilde{\varepsilon}_{p,r}^h \cdot p}{\mathbf{c}\,\mathbf{s}\,\xi_1}\right)\mathrm{Pf}'\,\Psi_{n-1}\ , \end{aligned} \tag{3.52}$$

since all other contributions in the Pfaffian expansion are proportional to ϵ. Note that Ψ_n with deleted first row/column will have the two equal rows/columns $(n+1)$ and $(n+2)$ in the collinear limit. The matrix Ψ_{n-1} is the matrix associated with the $(n-1)$ particle amplitude, i.e.

$$\Psi_{n-1} = \begin{bmatrix} 0 & A_{pb} & -C_{pp}^h & -C_{dp}^{h_d} \\ A_{ap} & A_{ab} & -C_{pa}^h & -C_{da}^{h_d} \\ C_{pp}^h & C_{pb}^h & 0 & B_{pd}^{h|h_d} \\ C_{cp}^{h_c} & C_{cb}^{h_c} & B_{cp}^{h_c|h} & B_{cd}^{h_c|h_d} \end{bmatrix}. \tag{3.53}$$

Opposite helicity case

Without loss of generality, we study the case $h_1 = +1$, $h_2 = -1$. If we compare this situation to the equal helicity case, we can spot two main differences. First, the matrix (3.49) will not have two equal rows/columns in the limit $\epsilon \to 0$. Second, the matrix is now carrying a divergence in the $B_{12} = \frac{1}{\epsilon\,\xi_1}$ component. Both complications are approached by additional matrix manipulations starting from 3.3.5, i.e.

$$\Psi_n = \begin{pmatrix} 0 & A_{12} & \epsilon\,\tilde{A}_{1b} & -\mathbf{s}^2\,C^+_{11} + \mathbf{c}^2\,C^+_{12} & \mathbf{c}^2\,C^-_{22} - \mathbf{s}^2\,C^-_{21} & -\epsilon\,\tilde{C}^{h_d}_{d1} \\ A_{21} & 0 & \tilde{A}_{2b} & -C^+_{12} - C^+_{11} & -C^-_{22} - C^-_{21} & -\tilde{C}^{h_d}_{d2} \\ \epsilon\,\tilde{A}_{a1} & \tilde{A}_{a2} & A_{ab} & -C^+_{1a} & -C^+_{2a} & -C^{h_d}_{da} \\ \mathbf{s}^2\,C^+_{11} - \mathbf{c}^2\,C^+_{12} & C^+_{12} + C^+_{11} & C^+_{1b} & 0 & B^{+|-}_{12} & B^{+|h_d}_{1d} \\ \mathbf{s}^2\,C^-_{21} - \mathbf{c}^2 C^-_{22} & C^-_{22} + C^-_{21} & C^-_{2b} & B^{-|+}_{21} & 0 & B^{-|h_d}_{2d} \\ \epsilon\,\tilde{C}^{h_c}_{c1} & \tilde{C}^{h_c}_{c2} & C^{h_c}_{cb} & B^{h_c|+}_{c1} & B^{h_c|-}_{c2} & B^{h_c|h_d}_{cd} \end{pmatrix} , \tag{3.54}$$

is further manipulated by

1. Add $\epsilon\,\mathbf{c}\,\mathbf{s}\,\tilde{\varepsilon}^-_{p,r}\cdot p$ times the $(n+1)$'st row/column to the first row/column ,
2. Add $\epsilon\,\mathbf{c}\,\mathbf{s}\,\tilde{\varepsilon}^+_{p,r}\cdot p$ times the $(n+2)$'nd row/column to the first row/column ,

while using the identities (we are using the short hand notations (3.28))

$$\begin{aligned} &\mathcal{Q}_{1,i} = (\tilde{\varepsilon}^+_{p,r}\cdot p)\ C^-_{pi} + (\tilde{\varepsilon}^-_{p,r}\cdot p)\ C^+_{pi} \quad , \quad \mathcal{Q}_1 = -\,(\tilde{\varepsilon}^+_{p,r}\cdot p)\ C^-_{pp} - \ (\tilde{\varepsilon}^-_{p,r}\cdot p)\ C^+_{pp} \ , \\ &C^+_{iq} = (\tilde{\varepsilon}^+_{p,r}\cdot p)\ B^{h_i|-}_{ip} \quad , \quad C^-_{iq} = (\tilde{\varepsilon}^-_{p,r}\cdot p)\ B^{h_i|+}_{ip} \ , \end{aligned} \tag{3.55}$$

which can be proven using spinor-helicity variables. We use (3.30) to express A_{12} in terms of $\mathcal{P}_2$ and $\mathcal{Q}$ arriving at (modulo higher order terms in ϵ)

$$\Big(0,\ \ -\epsilon\,\xi_1\,\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2,\ \ \epsilon\xi_1\mathbf{c}^2\mathbf{s}^2\,A^{(2)}_{pb},\ \ -\mathbf{s}^2 C^+_{11},\ \ \mathbf{c}^2 C^-_{22},\ \ -\epsilon\xi_1\mathbf{c}^2\mathbf{s}^2 C^{+,(2)}_{bp}\Big)\ . \tag{3.56}$$

Again we write

$$\mathrm{Pf}'\,\Psi_n = \mathrm{Pf}'^{(0)}\,\Psi_n + \epsilon\ \mathrm{Pf}'^{(1)}\,\Psi_n + \mathcal{O}(\epsilon^2)\ , \tag{3.57}$$

where now all terms contribute due to a $\frac{1}{\epsilon}$ term in the matrix, i.e.

$$\begin{aligned} \mathrm{Pf}'\,\Psi^{(0)}_n &= \frac{-1}{\sqrt{2}}\,\Big[\mathbf{s}^2\ \big(C^+_{pp} - C^+_{12}\big)\ \mathrm{Pf}'\,\Psi^{(1,n+1)}_n + \mathbf{c}^2\ \big(C^-_{pp} - C^-_{21}\big)\ \mathrm{Pf}'\,\Psi^{(1,n+2)}_n\Big] + \mathcal{G} \\ &= \frac{-1}{\sqrt{2}}\left[\mathbf{s}^2\left(C^+_{pp} - \frac{\mathbf{s}}{\mathbf{c}}\frac{2\,\tilde{\varepsilon}^+_{p,r}\cdot p}{\xi_1}\right)\mathrm{Pf}'\,\Psi^-_{n-1} + \mathbf{c}^2\left(C^-_{pp} - \frac{\mathbf{c}}{\mathbf{s}}\frac{2\,\tilde{\varepsilon}^-_{p,r}\cdot p}{\xi_1}\right)\mathrm{Pf}'\,\Psi^+_{n-1}\right] + \mathcal{G}\ . \end{aligned} \tag{3.58}$$

Here we defined $\Psi^-_{n-1} = \Psi^{(1,n+1)}_n$, $\Psi^+_{n-1} = \Psi^{(1,n+2)}_n$ as the matrices for a $(n-1)$ particle scattering where the particle with fused collinear momentum has either positive or negative helicity respectively. The object $\mathcal{G}$ contains all other terms of the expansion along the first row that are now $\mathcal{O}(1)$ due to the singular behavior of B_{12}. The important point is that $\mathcal{G}$ is independent of ξ_1 due to $B_{12} \propto \frac{1}{\xi_1}$.

3.4 Sum-over-solutions identities

After having established the relevant collinear expansions of the CHY building blocks, we need to determine the crucial expressions regarding sums over the two solutions of ξ_1. We express $\mathcal{J}_1$ trivially as a function of $\mathcal{J}_0$ and equivalently ξ_2 through ξ_1. We summarize

$$\sum_{\xi_1^\pm} \frac{\mathcal{J}_0^N}{\xi_1^M} = \left((x_1+\sqrt{x_2})^{N-M} + (-1)^N (x_1-\sqrt{x_2})^{N-M}\right) \frac{1}{(2\sqrt{x_2}h_1)^N} \ , \tag{3.59}$$

which follows from

$$\mathcal{J}_0\Big|_{\xi_1^\pm} = \frac{1}{2}\frac{1}{\underbrace{2\,\mathbf{s}^2\,\mathbf{c}^2\,\mathcal{P}_2}_{h_1} + \underbrace{(-\mathbf{cs}\mathcal{Q}_1)}_{h_2}\frac{1}{\xi_1^\pm}} = \frac{\xi_1^\pm}{h_1\,\xi_1^\pm + h_2} = \pm\frac{(x_1 \pm \sqrt{x_2})}{2\sqrt{x_2}h_1} \ , \tag{3.60}$$

where we used (3.30) and $h_1\,x_1 + h_2 = 0$. Further relevant identities read

$$\begin{aligned}
&\sum_{\xi_1} \mathcal{J}_0\,\xi_1^2 = \frac{\mathcal{Q}_1^2 + 2\,(p\cdot r)\,\mathcal{P}_2}{4\,\mathbf{c}^4\,\mathbf{s}^4\,\mathcal{P}_2^3} \quad , \quad \sum_{\xi_1} \mathcal{J}_0\,\xi_1 = \frac{\mathcal{Q}_1}{2\,\mathbf{c}^3\,\mathbf{s}^3\,\mathcal{P}_2} \quad , \quad \sum_{\xi_1} \mathcal{J}_0 = \frac{1}{2\,\mathbf{c}^2\,\mathbf{s}^2\mathcal{P}_2} \ , \\
&\sum_{\xi_1} \frac{\mathcal{J}_0}{\xi_1} = 0 \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0}{\xi_1^2} = \frac{1}{4\,p\cdot r} \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0}{\xi_1^3} = -\frac{\mathbf{c}\,\mathbf{s}\,\mathcal{Q}_1}{8\,(p\cdot r)^2} \ , \\
&\sum_{\xi_1} \frac{\mathcal{J}_0}{\xi_1^4} = \frac{\mathbf{c}^2\,\mathbf{s}^2\,\left(\mathcal{Q}_1^2 + 2\,(p\cdot r)\,\mathcal{P}_2\right)}{16\,(p\cdot r)^3} \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0^2}{\xi_1} = \frac{\mathcal{Q}_1}{4\,\mathbf{c}^3\,\mathbf{s}^3\,D\,\mathcal{P}_2} \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0^2}{\xi_1^2} = \frac{1}{2\,\mathbf{c}^2\,\mathbf{s}^2\,D} \ , \\
&\sum_{\xi_1} \frac{\mathcal{J}_0^2}{\xi_1^3} = -\frac{\mathcal{Q}_1}{8\,\mathbf{c}\,\mathbf{s}\,p\cdot r\,D} \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0^3}{\xi_1^4} = \frac{1}{16\,\mathbf{c}^2\,\mathbf{s}^2\,p\cdot r\,D} \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0^3}{\xi_1^5} = -\frac{\mathcal{Q}_1}{32\,\mathbf{c}\,\mathbf{s}\,(p\cdot r)^2\,D} \ ,
\end{aligned} \tag{3.61}$$

where we defined $D := \mathcal{Q}_1^2 + 8\,p\cdot r\,\mathcal{P}_2$. Finally, we have

$$\begin{aligned}
&\sum_{\xi_1} \frac{\mathcal{J}_0\xi_2}{\xi_1^2} = \frac{(\mathbf{c}^2-\mathbf{s}^2)}{4\,\mathbf{c}^s\,\mathbf{s}^2\,D}\left(4\,\mathcal{R}_1 - \frac{\mathcal{Q}_1\,\mathcal{Q}_2}{\mathcal{P}_2}\right) \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0\xi_2}{\xi_1^3} = -\frac{\mathbf{c}^2-\mathbf{s}^2}{2\,\mathbf{c}\,\mathbf{s}\,D}\left(\mathcal{Q}_2 + \frac{\mathcal{Q}_1\,\mathcal{R}_1}{2\,p\cdot r}\right) \ , \\
&\sum_{\xi_1} \frac{\mathcal{J}_0^2\xi_2}{\xi_1^4} = \frac{(\mathbf{c}^2-\mathbf{s}^2)\mathcal{R}_1}{8\,\mathbf{c}^2\,\mathbf{s}^2\,p\cdot r\,D} \quad , \quad \sum_{\xi_1} \frac{\mathcal{J}_0^2\xi_2}{\xi_1^5} = -\frac{\mathbf{c}^2-\mathbf{s}^2}{16\,\mathbf{c}\,\mathbf{s}\,p\cdot r\,D}\left(\frac{\mathcal{R}_1\,\mathcal{Q}_1}{p\cdot r} + \mathcal{Q}_2\right) \ .
\end{aligned} \tag{3.62}$$

3.5 Reproducing the Leading order Collinear Gluon Limit

Now we are ready to compute the divergent collinear contribution for YM partial amplitudes, checking whether our approach works. We will reproduce the leading behavior for both the equal and mixed helicity case but will focus on the equal helicity case for the sub-leading order. We expect to find the known universal behavior in terms of the helicity-dependent split function [57, 60], i.e.

$$\mathrm{A}_n^{(0)}(1^{h_1},2^{h_2},\ldots)=\sum_{h=\pm}\mathrm{Split}^{\mathrm{tree}}_{-h}(\mathbf{c};1^{h_1},2^{h_2})\,\mathrm{A}_{n-1}(p^h,\ldots)+\mathcal{O}(\epsilon^0)\ , \tag{3.63}$$

where the Split functions are traditionally given in spinor-helicity notation. The expressions read [60]

$$\begin{aligned}&\mathrm{Split}^{\mathrm{tree}}_{+}(\mathbf{c};1^+,2^+)=0\quad , && \mathrm{Split}^{\mathrm{tree}}_{+}(\mathbf{c};1^+,2^-)=-\frac{1}{\epsilon}\frac{\mathbf{s}^3}{\mathbf{c}\,\langle pr\rangle}\ ,\\ &\mathrm{Split}^{\mathrm{tree}}_{-}(\mathbf{c};1^+,2^+)=\frac{1}{\epsilon}\frac{1}{\mathbf{c}\,\mathbf{s}\,\langle pr\rangle}\quad , && \mathrm{Split}^{\mathrm{tree}}_{-}(\mathbf{c};1^+,2^-)=\frac{1}{\epsilon}\frac{\mathbf{c}^3}{\mathbf{s}\,[pr]}\ .\end{aligned} \tag{3.64}$$

3.5.1 Equal helicities ($h_1 = h_2 = +$)

Inserting the leading contributions of all building blocks and use the sum-over solution identities we obtain

$$\begin{aligned}\mathrm{A}_n^{(0)}&=-2\sum_{\xi_1}\int d\mu_{n-1}^{(0)}\,\frac{-\mathcal{J}_0}{\epsilon\,\xi_1}\,\mathfrak{C}_{n-1}\left(C^+_{pp}-\frac{2\,\tilde{\varepsilon}^+_{p,r}\cdot p}{s\,c\,\xi_1}\right)\mathrm{Pf}'\,\Psi_{n-1}\\ &=-\frac{2\,\tilde{\varepsilon}^+_{p,r}\cdot p}{\epsilon\,\mathbf{s}\,\mathbf{c}\,2\,p\cdot r}\int d\mu_{n-1}^{(0)}\,\mathfrak{C}_{n-1}\ \mathrm{Pf}'(\Psi_{n-1})=\frac{\sqrt{2}}{\epsilon\,\mathbf{c}\,\mathbf{s}\,\langle p\,r\rangle}\,\mathcal{A}_{n-1}\\ &=\sqrt{2}\,\mathrm{Split}^{\mathrm{tree}}_{-}\,\mathcal{A}_{n-1}\ ,\end{aligned} \tag{3.65}$$

which, up to a trivial normalization, is the correct behavior. If necessary, one can check the case $h_1 = h_2 = -$ straightforwardly.

3.5.2 Opposite helicities ($h_1 = +\ , h_2 = -$)

Due to $\sum_{\xi_1}\frac{\mathcal{J}_0}{\xi_1}=0$, the object $\mathcal{G}$ does not contribute and we easily obtain

$$\mathrm{A}_{n+2}^{(0)}(1^+,2^-,...)=\mathrm{Split}^{\mathrm{tree}}_{-}\mathcal{A}_{n-1}(P^+,...)+\mathrm{Split}^{\mathrm{tree}}_{+}\mathcal{A}_{n-1}(P^-,...)\ , \tag{3.66}$$

which is, up to a factor of $\sqrt{2}$, again the correct result.

3.6 The Sub-Leading Adjacent Collinear Gluon Limit

We now compute the sub-leading order contribution to the adjacent collinear gluon limit. Even though we are mainly interested in the equal helicity case, due to its connection to EYM amplitudes through ST-one (2.103), we will compute the expansion for any helicity configuration since it can be done effortless in the CHY formalism. We will introduce appropriate factors to keep track of both cases.

We divide the calculation into four terms, i.e.

$$\begin{aligned}\mathrm{A}_n^{(1)} = 2\sum_{\xi_1}\int d'\sigma_{n-2}\,d\sigma_p\Big[&\underbrace{\Delta_{n-1}^{(0)}\mathfrak{C}_n^{(1)}\mathcal{J}_0\,\mathrm{Pf}'^{(0)}\,\Psi_n}_{T_1} + \underbrace{\Delta_{n-1}^{(0)}\mathfrak{C}_n^{(0)}\mathcal{J}_1\,\mathrm{Pf}'^{(0)}\,\Psi_n}_{T_2} \\ &+ \underbrace{\Delta_{n-1}^{(0)}\mathfrak{C}_n^{(0)}\mathcal{J}_0\,\mathrm{Pf}'^{(1)}\,\Psi_n}_{T_3} + \underbrace{\Delta_{n-1}^{(1)}\mathfrak{C}_n^{(0)}\mathcal{J}_0\,\mathrm{Pf}'^{(0)}\,\Psi_n}_{T_4}\Big]\ ,\end{aligned} \tag{3.67}$$

and will now give the relevant structure of the terms in an equal- and mixed helicity adjustable way.

3.6.1 T_1: Parke-Taylor contribution

The relevant term to compute is

$$\sum_{\xi_1}\mathcal{J}_0\left(\frac{\xi_2}{\xi_1^2}+\frac{1}{2}S_{n,p,3}\right)\left(a\cdot C_{pp}^{h_p} - b\cdot\frac{2\,\tilde{\varepsilon}_{p,r}^{h_p}\cdot p}{\xi_1}\right)\ , \tag{3.68}$$

where the factors a and b can be adjusted depending on the helicity situation of the collinear gluons, i.e. whether one uses (3.52) or (3.58) for $\mathrm{Pf}'^{(0)}\,\Psi_n$. For $h_1 = h_2 = h$ we have $a = 1$ and $b = \frac{1}{\mathbf{s}\,\mathbf{c}}$ and for $h_1 \neq h_2$ this term appears twice with either $a = \mathbf{s}^2$ and $b = \frac{\mathbf{s}^3}{\mathbf{c}}$ or $a = \mathbf{c}^2$ and $b = \frac{\mathbf{c}^3}{\mathbf{s}}$. We evaluate this term using sum-over-solution identities (3.61) and (3.62).

$$\begin{aligned}&a\cdot\frac{(\mathbf{c}^s-\mathbf{s}^2)\,C_{pp}^{h_p}}{4\,\mathbf{c}^2\,\mathbf{s}^2\,D}\left(4\,\mathcal{R}_1 - \frac{\mathcal{Q}_1\,\mathcal{Q}_2}{\mathcal{P}_2}\right) + b\cdot\frac{(\mathbf{c}^2-\mathbf{s}^2)\,\tilde{\varepsilon}_{p,r}^{h_p}\cdot p}{\mathbf{c}\,\mathbf{s}\,D}\left(\mathcal{Q}_2 + \frac{\mathcal{Q}_1\,\mathcal{R}_1}{2\,p\cdot r}\right) \\ &+ a\cdot\frac{S_{n,p,3}\,C_{pp}^{h_p}}{4\,\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2}\ .\end{aligned} \tag{3.69}$$

3.6.2 T_2: Jacobian contribution

The relevant term is

$$\sum_{\xi_1}\frac{-1}{\xi_1}\mathcal{J}_0^2\left(8(p\cdot r)\frac{\xi_2}{\xi_1^3} - \mathbf{cs}(\mathbf{c}^2-\mathbf{s}^2)\mathcal{Q}_2\right)\left(a\cdot C_{pp}^{h_p} - b\cdot\frac{2\,\tilde{\varepsilon}_{p,r}^{h_p}\cdot p}{\xi_1}\right)\ , \tag{3.70}$$

which evaluates to

$$-a\cdot\frac{(\mathbf{c}^2-\mathbf{s}^2)\,C_{pp}^{h_p}}{4\,\mathbf{c}^2\,\mathbf{s}^2\,D}\left(4\,\mathcal{R}_1 - \frac{\mathcal{Q}_1\,\mathcal{Q}_2}{\mathcal{P}_2}\right) - b\cdot\frac{(\mathbf{c}^2-\mathbf{s}^2)\,\tilde{\varepsilon}_{p,r}^{h_p}\cdot p}{\mathbf{c}\,\mathbf{s}\,D}\left(2\mathcal{Q}_2 + \frac{\mathcal{Q}_1\,\mathcal{R}_1}{p\cdot r}\right)\ . \tag{3.71}$$

3.6.3 T_3: Pfaffian contribution.

Here we consider the sub-leading expansion of $\mathrm{Pf}'\,\Psi_n$ for an arbitrary helicity configuration of the collinear gluons after we performed the matrix manipulations from either the equal helicity case or the mixed helicity case. Then

$$
\begin{aligned}
\mathrm{Pf}'^{(1)}\,\Psi_n = &\frac{-1}{\sqrt{2}}\frac{\partial}{\partial\epsilon}\left(x\cdot\mathbf{s}^2\,C_{11}^{h_1}-y\cdot\mathbf{c}^2\,C_{12}^{h_1}\right)\Big|_{\epsilon=0}\,\mathrm{Pf}'\,\Psi_n^{(1,n+1)}\\
&-\frac{1}{\sqrt{2}}\frac{\partial}{\partial\epsilon}\left(x\cdot\mathbf{c}^2\,C_{22}^{h_2}-y\cdot\mathbf{s}^2\,C_{21}^{h_2}\right)\Big|_{\epsilon=0}\,\mathrm{Pf}'\,\Psi_n^{(1,n+2)}\\
&-\frac{1}{\sqrt{2}}\left(x\cdot\mathbf{s}^2\,C_{11}^{h_1}-y\cdot\mathbf{c}^2\,C_{12}^{h_1}\right)\frac{\partial}{\partial\epsilon}\,\mathrm{Pf}'\,\Psi_n^{(1,n+1)}\Big|_{\epsilon=0}\\
&-\frac{1}{\sqrt{2}}\left(x\cdot\mathbf{c}^2\,C_{22}^{h_2}-y\cdot\mathbf{s}^2\,C_{21}^{h_2}\right)\frac{\partial}{\partial\epsilon}\,\mathrm{Pf}'\,\Psi_n^{(1,n+2)}\Big|_{\epsilon=0}\,,
\end{aligned}
\tag{3.72}
$$

where we introduced the two numbers $x,\ y$ in order to quickly switch the manipulation scheme, i.e. $x=y=1$ for $h_1=h_2=h$ and $x=1,\ y=0$ for $h_1\neq h_2$. These are the only relevant terms of any sub-leading Pfaffian expansion of both (3.49) and (3.54) due to the equality of the lines $(n+1)$ and $(n+2)$ in the $h_1=h_2=h$ case or due to a higher order contribution from $\frac{\partial}{\partial\epsilon}\mathcal{G}$ (the reason is an involvement of higher orders in the ϵ expansion, i.e. $\xi=\epsilon\,\xi_1+\epsilon^2\,\xi_2+\epsilon^3\,\xi_3+\mathcal{O}(\epsilon^4)$, which is beyond the scope of the current discussion) in the mixed helicity case. Let us define the $(2n-2)\times(2n-2)$ matrices $\Psi_n^{(1,n+1)}:=\tilde{\Psi}$ and $\Psi_n^{(1,n+2)}:=\hat{\Psi}$ for the sake of a compact notation.
The contribution to the sub-leading order of the first two terms in (3.72) are quickly evaluated to be

$$
\begin{aligned}
&\sum_{\xi_1}\frac{-\mathcal{J}_0}{\xi_1}\frac{\partial}{\partial\epsilon}\left(x\cdot\mathbf{s}^2\,C_{11}^{h_1}-y\cdot\mathbf{c}^2\,C_{12}^{h_1}\right)\Big|_{\epsilon=0}=\\
&=-\left(x\cdot\mathbf{s}^2\frac{C_{pp}^{h_1,(2)}}{4\,\mathbf{c}^2\mathbf{s}^2\,\mathcal{P}_2}-(x\cdot\mathbf{s}^2+y\cdot\mathbf{c}^2)\frac{\mathbf{s}}{\mathbf{c}}\,\tilde{\varepsilon}_{p,r}^{h_1}\cdot p\frac{(\mathbf{c}^2-\mathbf{s}^2)}{\mathbf{s}\,\mathbf{c}\,D}\left(\mathcal{Q}_2+\frac{\mathcal{Q}_1\,\mathcal{R}_1}{2\,p\cdot r}\right)\right)
\end{aligned}
\tag{3.73}
$$

$$
\begin{aligned}
&\sum_{\xi_1}\frac{-\mathcal{J}_0}{\xi_1}\frac{\partial}{\partial\epsilon}\left(x\cdot\mathbf{c}^2\,C_{22}^{h_2}-y\cdot\mathbf{s}^2\,C_{21}^{h_2}\right)\Big|_{\epsilon=0}=\\
&=\left(x\cdot\mathbf{c}^2\frac{C_{pp}^{h_1,(2)}}{4\,\mathbf{c}^2\mathbf{s}^2\,\mathcal{P}_2}+(x\cdot\mathbf{c}^2+y\cdot\mathbf{s}^2)\frac{\mathbf{c}}{\mathbf{s}}\,\tilde{\varepsilon}_{p,r}^{h_2}\cdot p\frac{(\mathbf{c}^2-\mathbf{s}^2)}{\mathbf{s}\,\mathbf{c}\,D}\left(\mathcal{Q}_2+\frac{\mathcal{Q}_1\,\mathcal{R}_1}{2\,p\cdot r}\right)\right)\ .
\end{aligned}
\tag{3.74}
$$

For the last two terms we directly apply (3.48) without regarding the normalization, i.e.

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon}\,\mathrm{Pf}'\,\tilde{\Psi} = \; & \underbrace{\sum_{i=2}^{2\,n-2} (-1)^{i}\,\frac{\partial\,\tilde{\Psi}_{1,i}}{\partial\,\epsilon}\,\mathrm{Pf}'\,\tilde{\Psi}^{1,i}}_{\text{Terms with } \epsilon \text{ dependence in first row}} \;+\; \underbrace{\sum_{i=2}^{n-1} (-1)^{n+i+1}\,\frac{\partial\,\tilde{\Psi}_{i,n}}{\partial\,\epsilon}\,\mathrm{Pf}'\,\tilde{\Psi}^{i,n}}_{\text{Terms with } \epsilon \text{ dependence in } n'th \text{ column}} \\
& + \underbrace{\sum_{i=n+1}^{2n-2} (-1)^{n+i+1}\,\frac{\partial\,\tilde{\Psi}_{n,i}}{\partial\,\epsilon}\,\mathrm{Pf}'\,\tilde{\Psi}^{n,i}}_{\text{Terms with } \epsilon \text{ dependence in } n'th \text{ row}} + \underbrace{\sum_{i=n+1}^{2n-2} (-1)^{n}\,\frac{\partial\,\tilde{\Psi}_{n-i+3,i}}{\partial\,\epsilon}\,\mathrm{Pf}'\,\tilde{\Psi}^{n-i+3,i}}_{\substack{\text{Terms with } \epsilon \text{ dependence in the diagonal of } C \\ \text{not including } C_{11}}}\;.
\end{aligned}
\tag{3.75}
$$

The same formula holds true for $\hat{\Psi}$. We identify

$$
\begin{aligned}
&\tilde{\Psi}_{1,i} = \begin{cases} \tilde{A}_{2,i+1} & i \le n-1 \\ -\tilde{C}^{h_2}_{i-n+2,2} & i \ge n+1 \\ -C^{h_2}_{22} - C^{h_2}_{21} & i = n \end{cases} \quad , \qquad \hat{\Psi}_{1,i} = \begin{cases} \tilde{A}_{2,i+1} & i \le n-1 \\ -\tilde{C}^{h_1}_{i-n+2,1} & i \ge n+1 \\ -C^{h_1}_{11} - C^{h_1}_{12} & i = n \end{cases} \;, \\
&\tilde{\Psi}_{i,n} = -C^{h_2}_{2,i+1} \quad , \qquad \tilde{\Psi}_{n,i} = B^{h_2|h_{i-n+2}}_{2,i-n+2} \quad , \qquad \tilde{\Psi}_{n-i+3,i} = -C^{h_{i-n+2}}_{i-n+2,i-n+2}\;, \\
&\hat{\Psi}_{i,n} = -C^{h_1}_{1,i+1} \quad , \qquad \hat{\Psi}_{n,i} = B^{h_1|h_{i-n+2}}_{1,i-n+2} \quad , \qquad \hat{\Psi}_{n-i+3,i} = -C^{h_{i-n+2}}_{i-n+2,i-n+2}\;.
\end{aligned}
\tag{3.76}
$$

The rest of the calculation is straightforward. Taking into account the ξ_1 dependence of $\mathfrak{C}^{(0)}_n$ and $\mathcal{J}_0$, we compute

$$
\begin{aligned}
&\frac{-1}{\sqrt{2}}\sum_{\xi_1}\frac{-\mathcal{J}_0}{\xi_1}\,(x\cdot\mathbf{s}^2\,C^{h_1}_{11} - y\cdot\mathbf{c}^2\,C^{h_1}_{12})\,\frac{\partial}{\partial\epsilon}\,\mathrm{Pf}'\,\Psi^{(1,n+1)}_{n}\Big|_{\epsilon=0} = \\
&\frac{1}{\sqrt{2}}\,\frac{x\cdot\mathbf{s}^2\,C^{h_1}_{pp}}{4\,\mathbf{c}^s\,\mathbf{s}^2\,\mathcal{P}_2}\Big\{\sum_{i=2}^{n-1}(-1)^i\,(\mathbf{c}^2-\mathbf{s}^2)\,A^{(2)}_{p,i+1}\,\mathrm{Pf}'\,\tilde{\Psi}^{1,i} \\
&+\sum_{i=n+1}^{2n-2}(-1)^i\,(\mathbf{c}^2-\mathbf{s}^2)\,C^{h_{i-n+2},(2)}_{i-n+2,p}\,\mathrm{Pf}'\,\tilde{\Psi}^{1,i} + (-1)^n\,C^{h_2,(2)}_{pp}\,\mathrm{Pf}'\,\tilde{\Psi}^{1,n} \\
&+\sum_{i=n+1}^{2n-2}(-1)^{n+i}\,B^{h_2|h_{i-n+2},(2)}_{p,i-n+2}\,\mathrm{Pf}'\,\tilde{\Psi}^{n,i} - \sum_{i=2}^{n-1}(-1)^{n+i}\,C^{h_2,(2)}_{p,i+1}\,\mathrm{Pf}'\,\tilde{\Psi}^{n,i} \\
&-\sum_{i=n+1}^{2n-2}(-1)^n\,(\mathbf{c}^2-\mathbf{s}^2)\,C^{h_{i-n+2},(2)}_{i-n+2,p}\,\mathrm{Pf}'\,\tilde{\Psi}^{n-i+3,i}\Big\} \\
&+\frac{(-1)^{n+1}}{\sqrt{2}}\,2\,(x\cdot\mathbf{s}^2+y\cdot\mathbf{c}^2)\,\tilde{\varepsilon}^{h_1}_{p,r}\cdot p\,E^{h_2}\,\mathrm{Pf}'\,\tilde{\Psi}^{1,n}\;,
\end{aligned}
\tag{3.77}
$$

and

$$
\begin{aligned}
&\frac{-1}{\sqrt{2}} \sum_{\xi_1} \frac{-\mathcal{J}_0}{\xi_1} (x \cdot \mathbf{c}^2\, C_{22}^{h_2} - y \cdot \mathbf{s}^2\, C_{21}^{h_2}) \frac{\partial}{\partial \epsilon} \operatorname{Pf}'\, \Psi_n^{(1,n+2)}\Big|_{\epsilon=0} = \\
&\frac{1}{\sqrt{2}} \frac{x \cdot \mathbf{c}^2\, C_{pp}^{h_2}}{4\, \mathbf{c}^s\, \mathbf{s}^2\, \mathcal{P}_2} \Big\{ \sum_{i=2}^{n-1} (-1)^i\, (\mathbf{c}^2 - \mathbf{s}^2)\, A_{p,i+1}^{(2)} \operatorname{Pf}'\, \hat{\Psi}^{1,i} \\
&+ \sum_{i=n+1}^{2n-2} (-1)^i\, (\mathbf{c}^2 - \mathbf{s}^2)\, C_{i-n+2,p}^{h_{i-n+2},(2)} \operatorname{Pf}'\, \hat{\Psi}^{1,i} - (-1)^n\, C_{pp}^{h_1,(2)} \operatorname{Pf}'\, \hat{\Psi}^{1,n} \\
&- \sum_{i=n+1}^{2n-2} (-1)^{n+i}\, B_{p,i-n+2}^{h_1|h_{i-n+2},(2)} \operatorname{Pf}'\, \hat{\Psi}^{n,i} + \sum_{i=2}^{n-1} (-1)^{n+i}\, C_{p,i+1}^{h_1,(2)} \operatorname{Pf}'\, \hat{\Psi}^{n,i} \\
&- \sum_{i=n+1}^{2n-2} (-1)^n\, (\mathbf{c}^2 - \mathbf{s}^2)\, C_{i-n+2,p}^{h_{i-n+2},(2)} \operatorname{Pf}'\, \hat{\Psi}^{n-i+3,i} \Big\} \\
&- \frac{(-1)^{n+1}}{\sqrt{2}}\, 2\, (x \cdot \mathbf{s}^2 + y \cdot \mathbf{c}^2)\, \tilde{\varepsilon}_{p,r}^{h_2} \cdot p\, E^{h_1} \operatorname{Pf}'\, \tilde{\Psi}^{1,n} \; .
\end{aligned}
\tag{3.78}
$$

We can compare both terms with

$$
\begin{aligned}
&\frac{\partial}{\partial\, \sigma_p} \operatorname{Pf}'^{\pm}\, \Psi_{n-1} = \sum_{i=2}^{n-1} (-1)^{i+1}\, A_{p,i+1}^{(2)} \operatorname{Pf}'^{\pm}\, \Psi_{n-1}^{1,i} \\
&+ \sum_{i=n+1}^{2n-2} (-1)^{i+1}\, C_{i-n+2,p}^{h_{i-n+2},(2)} \operatorname{Pf}'^{\pm}\, \Psi_{n-1}^{1,i} + (-1)^n\, C_{pp}^{\pm,(2)} \operatorname{Pf}'^{\pm}\, \Psi_{n-1}^{1,n} \\
&+ \sum_{i=n+1}^{2n-2} (-1)^{n+i}\, B_{p,i-n+2}^{\pm|h_{i-n+2},(2)} \operatorname{Pf}'^{\pm}\, \Psi_{n-1}^{n,i} + \sum_{i=2}^{n-1} (-1)^{n+i+1}\, C_{p,i+1}^{\pm,(2)} \operatorname{Pf}'^{\pm}\, \Psi_{n-1}^{n,i} \\
&+ \sum_{i=n+1}^{2n-2} (-1)^n\, C_{i-n+2,p}^{h_{i-n+2},(2)} \operatorname{Pf}'^{\pm}\, \Psi_{n-1}^{n-i+3,i} \; ,
\end{aligned}
\tag{3.79}
$$

to see that summing (3.77) and (3.78) in the case of $h_1 = h_2 = h$ ($x = y = 1$) yields

$$
\begin{aligned}
&\frac{-1}{\sqrt{2}} \sum_{\xi_1} \frac{-\mathcal{J}_0}{\xi_1} (\mathbf{s}^2\, C_{11}^{h} - \mathbf{c}^2\, C_{12}^{h}) \frac{\partial}{\partial \epsilon} \operatorname{Pf}'\, \Psi_n^{(1,n+1)}\Big|_{\epsilon=0} \\
&- \frac{1}{\sqrt{2}} \sum_{\xi_1} \frac{-\mathcal{J}_0}{\xi_1} (\mathbf{c}^2\, C_{22}^{h} - \mathbf{c}^2\, C_{21}^{h}) \frac{\partial}{\partial \epsilon} \operatorname{Pf}'\, \Psi_n^{(1,n+2)}\Big|_{\epsilon=0} \\
&= \frac{-1}{\sqrt{2}} \frac{(\mathbf{c}^2 - \mathbf{s}^2)\, C_{pp}^{h}}{4\, \mathbf{c}^2\, \mathbf{s}^2\, \mathcal{P}_2} \frac{\partial}{\partial\, \sigma_p} \operatorname{Pf}'^{\pm}\, \Psi_{n-1} \; .
\end{aligned}
\tag{3.80}
$$

We refrain from writing out the easily obtainable expression for $h_1 \neq h_2$.

3.6.4 T_4: δ-functions contribution

The relevant term is

$$\sum_{\xi_1} \frac{-\mathcal{J}_0}{\xi_1}\, d\,\mu^{(1)}_{n-1} \left(a \cdot C^{h_p}_{pp} - b \cdot \frac{2\,\tilde{\varepsilon}^{h_p}_{p,r} \cdot p}{\xi_1} \right) = d'\sigma_{n-1}\, \frac{a \cdot (\mathbf{c}^2 - \mathbf{s}^2)\, C^{h_p}_{pp}}{4\,\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2}\, \frac{\partial}{\partial\,\sigma_p}\, \Delta'_{n-1}\ , \tag{3.81}$$

which can be quickly seen by direct computation.

3.6.5 Full result

Combining everything yields the final result for equal helicity collinear gluons

$$\begin{aligned} \mathrm{A}^{(1)}_n = &- \int d\mu^{(0)}_{n-1} \left(\frac{C^h_{pp}}{2\,\mathbf{s}^2\,\mathbf{c}^2\,\mathcal{P}_2} S(n,p,3) + \frac{\mathbf{c}^2 - \mathbf{s}^2}{2\,\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2}\, C^{h,(2)}_{pp} \right) \mathfrak{C}_{n-1}\ \mathrm{Pf}'\,\Psi_{n-1} \\ &- \frac{\mathbf{c}^2 - \mathbf{s}^2}{2\,\mathbf{c}^2\,\mathbf{s}^2} \int d\mu^{(0)}_{n-1}\, \frac{C^h_{pp}}{\mathcal{P}_2}\, \mathfrak{C}_{n-1}\, \frac{\partial}{\partial\sigma_p} \left(\mathrm{Pf}'\,\Psi_{n-1} \Delta'^{(0)}_{n-1} \right)\ . \end{aligned} \tag{3.82}$$

We can further proceed by using partial integration

$$\begin{aligned} \mathrm{A}^{(1)}_n = &- \int d\mu^{(0)}_{n-1} \Bigg(\frac{C^h_{pp}}{2\,\mathbf{s}^2\,\mathbf{c}^2\,\mathcal{P}_2} S(n,p,3) + \frac{\mathbf{c}^2 - \mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2} \left(C^{h,(2)}_{pp} - \frac{C^h_{pp}\mathcal{P}_3}{\mathcal{P}_2} \right) \\ &- \frac{\mathbf{c}^2 - \mathbf{s}^2}{2\,\mathbf{c}^2\,\mathbf{s}^2}\, \frac{C^h_{pp}}{\mathcal{P}_2} \left(S_{n,p,3} - \frac{2}{\sigma_{p3}} \right) \Bigg) \mathfrak{C}_{n-1}\ \mathrm{Pf}'\,\Psi_{n-1} \\ &- \frac{\mathbf{c}^2 - \mathbf{s}^2}{2\,\mathbf{c}^2\,\mathbf{s}^2} \int d'\sigma_{n-2}\, d\sigma_p\, \frac{\partial}{\partial\sigma_p} \left(\frac{C^h_{pp}}{\mathcal{P}_2}\, \mathfrak{C}_{n-1}\ \mathrm{Pf}'\,\Psi_{n-1} \Delta'^{(0)}_{n-1} \right)\ , \end{aligned} \tag{3.83}$$

which may be rewritten as

$$\begin{aligned} \mathrm{A}^{(1)}_n = &- \int d\mu^{(0)}_{n-1} \left(\frac{C^h_{pp}}{\mathcal{P}_2} \left(\frac{1}{\mathbf{c}^2}\,\frac{1}{\sigma_{np}} + \frac{1}{\mathbf{s}^2}\,\frac{1}{\sigma_{p3}} \right) + \frac{\mathbf{c}^2 - \mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2} \left(C^{h,(2)}_{pp} - \frac{C^h_{pp}\mathcal{P}_3}{\mathcal{P}_2} \right) \right) \mathfrak{C}_{n-1}\ \mathrm{Pf}'\,\Psi_{n-1} \\ &- \frac{\mathbf{c}^2 - \mathbf{s}^2}{2\,\mathbf{c}^2\,\mathbf{s}^2} \int d'\sigma_{n-2}\, d\sigma_p\, \frac{\partial}{\partial\sigma_p} \left(\frac{C^h_{pp}}{\mathcal{P}_2}\, \mathfrak{C}_{n-1}\ \mathrm{Pf}'\,\Psi_{n-1} \Delta'^{(0)}_{n-1} \right)\ . \end{aligned} \tag{3.84}$$

The last term is a total derivative w.r.t σ_p and can be dropped. Furthermore, we define the collinear gluon kernel by

$$\mathcal{K}^{\text{gluon}}_{\text{coll}}(\varepsilon^h_p, \{p_i\}, \{\sigma_i\}) = - \left(\frac{C^h_{pp}}{\mathcal{P}_2} \left(\frac{1}{\mathbf{c}^2}\,\frac{1}{\sigma_{np}} + \frac{1}{\mathbf{s}^2}\,\frac{1}{\sigma_{p3}} \right) + \frac{\mathbf{c}^2 - \mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2} \left(C^{h,(2)}_{pp} - \frac{C_{pp}\mathcal{P}_3}{\mathcal{P}_2} \right) \right)\ , \tag{3.85}$$

s.t. we can write the final result compactly as

$$\mathrm{A}^{(1)}_n = \int d\mu^{(0)}_{n-1}\, \mathcal{K}^{\text{gluon}}_{\text{coll}}(\varepsilon^h_p, \{p_i\}, \{\sigma_i\})\, \mathcal{I}^{\text{Yang-Mills}}_{n-1}\ . \tag{3.86}$$

The kernel $\mathcal{K}^{\text{gluon}}_{\text{coll}}(\varepsilon_p^h, \{p_i\}, \{\sigma_i\})$ reflects a factorization at the level of the CHY integrand.

Although (3.86) is a compact expression, no factorized structure seems to be achievable at the level of the amplitude. The reason is mainly the appearance of $\mathcal{P}_2$ in the denominator. We will shortly see and prove that it is impossible to obtain a factorization. But first, let us turn to the proof of universality.

3.7 Universality of the sub-leading collinear gluon limit

In this section we want to proof universality of our result (3.84). Note that our CHY approach only allows for an analysis of universality in the presence of massless bi-adjoins scalars and gravitons, since those are the only building blocks available. We will perform the proof in a straightforward manner, i.e. we will replace the CHY building blocks such that gluons are replaced by either scalars or gravitons. Then we will simply show that the collinear kernel (3.85) does not change.

3.7.1 Gluons → gravitons or scalars

We focus on the EYM integrand given in (2.119) with k gravitons and $n-k$ gluons with collinear gluons $1, 2$ and particles $3, n$ being gluons s.t. the structure (3.41) is not changed. The sub-leading order of the amplitude is computed via

$$\begin{aligned} \mathrm{A}_n^{(1)} = 2 \sum_{\xi_1} \int d'\sigma_{n-2}\, d\sigma_p \Big[& \underbrace{\Delta^{(0)}_{n-1} \mathfrak{C}^{(1)}_{n-k} \mathcal{J}_0 \ \mathrm{Pf}^{(0)}\, \Psi_k \ \mathrm{Pf}'^{(0)}\, \Psi_n}_{T_1} \\ & + \underbrace{\Delta^{(0)}_{n-1} \mathfrak{C}^{(0)}_{n-k} \mathcal{J}_1 \ \mathrm{Pf}^{(0)}\, \Psi_k \ \mathrm{Pf}'^{(0)}\, \Psi_n}_{T_2} + \underbrace{\Delta^{(0)}_{n-1} \mathfrak{C}^{(0)}_{n-k} \mathcal{J}_0 \ \mathrm{Pf}^{(0)}\, \Psi_k \ \mathrm{Pf}'^{(1)}\, \Psi_n}_{T_3} \\ & + \underbrace{\Delta^{(1)}_{n-1} \mathfrak{C}^{(0)}_{n-k} \mathcal{J}_0 \ \mathrm{Pf}^{(0)}\, \Psi_k \ \mathrm{Pf}'^{(0)}\, \Psi_n}_{T_4} + \underbrace{\Delta^{(0)}_{n-1} \mathfrak{C}^{(0)}_{n-k} \mathcal{J}_0 \ \mathrm{Pf}^{(1)}\, \Psi_k \ \mathrm{Pf}'^{(0)}\, \Psi_n}_{T_5} \Big] . \end{aligned} \tag{3.87}$$

We claim that this yields (3.84) for $h_1 = h_2 = h$ with the replacements $\mathfrak{C}_{n-1} \to \mathfrak{C}_{n-k-1}$ and $\mathrm{Pf}'\, \Psi_{n-1} \to \mathrm{Pf}\, \Psi_k \ \mathrm{Pf}'\, \Psi_{n-1}$. In order to prove this claim we only need to show that the last term can be written in terms of a σ_p derivative acting on the graviton Pfaffian. This is easy to see, since the only place that the graviton Pfaffian has any ϵ dependence are the diagonal entries of the C matrix which is the last term in (3.75) but this time also including $C_{n-k+1,n-k+1}$, i.e. the first diagonal element in C. Repeating the same computation as in the previous section for $h_1 = h_2 = h$, we see that

$$\mathrm{Pf}^{(1)}\, \Psi_k = (\mathbf{c}^2 - \mathbf{s}^2)\, \frac{\xi_1}{2} \sum_{i=k+1}^{2k} C^{h_{n-i+2},(2)}_{n-i+2,p} \ \mathrm{Pf}\, \Psi_k^{k-i+2,i} . \tag{3.88}$$

We thus indeed get

$$\frac{-1}{\sqrt{2}} \sum_{\xi_1} \frac{-\mathcal{J}_0}{\xi_1} \ \mathrm{Pf}^{(0)}\, \Psi_k \ \mathrm{Pf}'^{(0)}\, \Psi_n = -\frac{1}{\sqrt{2}} \frac{(\mathbf{c}^2 - \mathbf{s}^2)\, C^h_{pp}}{4\, \mathbf{c}^2\, \mathbf{s}^2\, \mathcal{P}_2} \ \mathrm{Pf}'\, \Psi_{n-1} \frac{\partial}{\partial \sigma_p} \mathrm{Pf}\, \Psi_k . \tag{3.89}$$

Hence T_5 only contributes to the neglectable total derivative term in (3.84). The collinear kernel (3.85) remains unchanged. Changing gluons into scalars also cannot have any effect on the structure of (3.84) since the only thing that will happen is that the size of the matrix Ψ_n will change into Ψ_{n-k} for k scalars and some parts of $\mathfrak{C}_n$ will now be squared. But those parts have no ϵ dependence since particles $1, 2, 3, n$ are fixed to be gluons. Therefore (3.84) is true up to an adjustment of $\mathfrak{C}_{n-1}$ and $\mathrm{Pf}'\,\Psi_{n-1} \to \mathrm{Pf}'\,\Psi_{n-k-1}$. Thus we conclude that the collinear kernel (3.85) is universal.

3.8 Two intriguing observations regarding factorization

As we have mentioned, our final result (3.84) is CHY universal, i.e. in the presence of gravitons and bi-adjoint scalars, but we were not able to find any factorization property at the level of the amplitude. We might ask whether this is the case because no factorization exists or because we are merely unable. The following observations shall shed light into this issue.

First, it is very interesting to remark that the final expression (3.84) can be written as

$$\mathrm{A}_n^{(1)} = \varepsilon_p^{\mu}\,\varepsilon_p^{\nu}\,\mathrm{A}_{n,\mu\nu}^{(1)} \ , \tag{3.90}$$

i.e. if we compare this to our schematic analysis in the beginning of this chapter (3.4) we can conclude that

$$\left(\varepsilon_p^{\mu}\,\delta\varepsilon_{i+1}^{\nu} + \delta\varepsilon_i^{\mu}\,\varepsilon_p^{\nu}\right)\mathrm{A}_{n,\mu\nu}^{(0)} = 0 \ . \tag{3.91}$$

A rather unexpected and unintuitive observation.

Second, we can ask for gauge invariance in the fused momentum p. Doing so in the explicit expression (3.84) reveals

$$p \cdot \frac{\partial}{\partial\,\varepsilon_p^h}\,\mathrm{A}_n^{(1)} = \frac{\mathbf{c}^2 - \mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2}\,\mathrm{A}_{n-1}^{\mathrm{YM}}(p, 3, ..., n) \ , \tag{3.92}$$

i.e. a gauge variation yields a lower point pure YM amplitude where the fused momentum and polarization now belong to a gluon. The analytic reason for this relation follows from the identities

$$p \cdot \frac{\partial}{\partial\,\varepsilon_p^h}\,C_{pp}^h = \mathcal{P}_1 = 0 \quad , \quad p \cdot \frac{\partial}{\partial\,\varepsilon_p^h} C_{pp}^{h,(2)} = \mathcal{P}_2 \quad , \quad p \cdot \frac{\partial}{\partial\,\varepsilon_p^h}\,\mathcal{I}_{n-1}^{\text{Yang-Mills}} = 0 \ , \tag{3.93}$$

where the first equality holds on the support of the scattering equations. We may ask why a gauge variation produces a lower point YM amplitude. The remarkable answer lies in ST-one (2.103), i.e.

$$\frac{\kappa\,x}{\mathrm{g}^2} \sum_{\rho \in S_{n-3}} (-1)^{m_\rho}\, s_{\rho(n-2)\,p}\,\mathrm{A}_n^{\mathrm{YM}}(1, \rho(2), ..., \rho(n-2), n-1, n) \; = \mathrm{A}_{n-1}^{\mathrm{EYM}}(1, ..., n-2; p) \ . \tag{3.94}$$

Taking the gauge variation of the above formula directly yields

$$\frac{\mathbf{c}^2 - \mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2} \sum_{\rho \in S_{n-3}} (-1)^{m_\rho}\, s_{\rho(n-2)\,p}\,\mathrm{A}_{n-1}^{\mathrm{YM}}(1, \rho(2), ..., \rho(n-2), p) \; = 0 \ , \tag{3.95}$$

due to the trivial gauge invariance of the EYM amplitude. As we have already seen, the above sum is isomorphic to BCJ relations (2.106) if we plug in lower point amplitudes, which is exactly the reason why the divergent part of the collinear limit does not contribute to ST-one. Therefore we find BCJ relations. A highly intriguing result. We conclude that the gauge variation of (3.84) is lower point YM amplitude such that gauge invariance of the EYM amplitude in ST-one is guaranteed.

3.9 Proving Stieberger-Taylor Identities

In this section we shall proof ST-one (2.103). We begin with a straightforward rewriting of $\mathcal{P}_2$ (3.28) in terms of insertion operators (2.123), i.e.

$$\mathcal{P}_2 = \sum_{a=1}^{n-1} p \cdot X_a \, S(a,p,a+1) \left[\frac{1}{\sigma_{ap}} - \frac{1}{\sigma_{p\,a+1}} \right] \;, \tag{3.96}$$

with region momentum $X_a = \sum_{b=1}^{a} p_b$ and. We can easily proof this statement

$$\begin{aligned}
&\sum_{a=1}^{n-1} p \cdot X_a \, S(a,p,a+1) \left[\frac{1}{\sigma_{ap}} - \frac{1}{\sigma_{p\,a+1}} \right] \\
&= \sum_{a=1}^{n-1} p \cdot X_a \left[\frac{1}{\sigma_{ap}} + \frac{1}{\sigma_{p\,a+1}} \right] \left[\frac{1}{\sigma_{ap}} - \frac{1}{\sigma_{p\,a+1}} \right] \\
&= \sum_{a=1}^{n-1} p \cdot X_a \left[\frac{1}{\sigma_{ap}^2} - \frac{1}{\sigma_{p\,a+1}^2} \right] \\
&= \sum_{a=1}^{n-1} p \cdot X_a \, \frac{1}{\sigma_{ap}^2} - \sum_{a=2}^{n} p \cdot X_{a-1} \, \frac{1}{\sigma_{ap}^2} \\
&= \sum_{a=1}^{n} \frac{p \cdot p_a}{\sigma_{ap}^2} = \mathcal{P}_2 \;,
\end{aligned} \tag{3.97}$$

where we have used the definition of the insertion operator in the second equality and additionally momentum conservation and on-shell conditions.

Now consider the object

$$\left[\frac{1}{\mathbf{c}^2} \frac{1}{\sigma_{ap}} + \frac{1}{\mathbf{s}^2} \frac{1}{\sigma_{p\,a+1}} \right] S(a,p,a+1) = F(a,p,a+1) \;. \tag{3.98}$$

It is easy to see that

$$\sum_{a=1}^{n-1} p \cdot X_a \, \left[F(a,p,a+1) - F(a+1,p,a) \right] = \frac{\mathbf{s}^2 - \mathbf{c}^2}{\mathbf{s}^2 \, \mathbf{c}^2} \, \mathcal{P}_2 \;. \tag{3.99}$$

Now let us turn to our final result (3.84) while rewriting

$$\begin{aligned}\mathfrak{C}_{n-1} &= \frac{1}{\sigma_{p3}\sigma_{34}\cdots\sigma_{np}} \\ &= \frac{\sigma_{n3}}{\sigma_{p3}\sigma_{np}}\,\frac{1}{\sigma_{34}\sigma_{45}\cdots\sigma_{n3}} \\ &= S(n,p,3)\,\mathfrak{C}_{n-2}\ ,\end{aligned} \tag{3.100}$$

such that we have no σ_p dependence left in the Parke-Taylor factor $\mathfrak{C}_{n-2}$. Using our new notion of $F(a,p,a+1)$ we can rewrite the sub-leading collinear result as

$$\mathrm{A}_n^{(1)} = -\int d\mu_{n-1}^{(0)}\left(\frac{C_{pp}^h}{\mathcal{P}_2}\,F(n,p,3) + \Lambda\, S(n,p,3)\right)\mathfrak{C}_{n-2}\ \mathrm{Pf}'\,\Psi_{n-1}\ , \tag{3.101}$$

where we have introduced the permutation invariant object

$$\Lambda = \frac{\mathbf{c}^2-\mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2\,\mathcal{P}_2}\left(C_{pp}^{h,(2)} - \frac{C_{pp}^h\mathcal{P}_3}{\mathcal{P}_2}\right)\ , \tag{3.102}$$

for compactness. Now let us switch to a more advantageous notation. First, we want to switch from the specified indices of the collinear gluon pair to a generic index pair, i.e.

$$p_1||p_2 \to p_A||p_B\ . \tag{3.103}$$

Second, we want to position the adjacent collinear gluon pair at an arbitrary position in the arguments of the amplitude. For that we consider a generic n-point YM amplitude $\mathrm{A}_n^{\mathrm{YM}}(1,...,n)$ with canonical ordering. Then we insert the adjacent collinear pair somewhere in between two indices a and $a+1$ arriving at an $(n+2)$-point YM amplitude. Finally, we perform the collinear limit in the same fashion as before, arriving at the sub-leading contribution of the amplitude

$$\mathrm{A}_{n+2}^{\mathrm{p_A}||\mathrm{p_B}}(1,...,a,A,B,a+1,...,n)\ , \tag{3.104}$$

i.e.

$$\begin{aligned}\mathrm{A}_{n+2}^{(1)}(1,...,a,p,a+1,...,n) = -\int d\mu_{n+1}^{(0)}\Bigg(&\frac{C_{pp}^h}{\mathcal{P}_2}\,F(a,p,a+1) \\ &+\Lambda\,S(a,p,a+1)\Bigg)\mathfrak{C}_n(1,...,n)\ \mathrm{Pf}'\,\Psi_{n+1},\end{aligned} \tag{3.105}$$

where we now put emphasis on the position of the fused momentum p. All other definitions and notations are unchanged. Now define the mirrored version of $\mathrm{A}_{n+2}^{(1)}(1,...,a,p,a+1,...,n)$ by $\mathrm{A}_{n+2}^{(1)}(n,...,a+1,p,a,...,1)$ where the sequence of arguments is reversed. Using this notation, we can evaluate the linear combination

$$\begin{aligned}&\mathrm{A}_{n+1}^{(1)}(1,...,a,p,a+1,...,n) - (-1)^n\,\mathrm{A}_{n+1}^{(1)}(n,...,a+1,p,a,...,1) \\ &= -\int d\mu_{n+1}^{(0)}\Bigg(\frac{C_{pp}^h}{\mathcal{P}_2}\,[F(a,p,a+1)-F(a+1,p,a)] \\ &\qquad +2\,\Lambda\,S(a,p,a+1)\Bigg)\mathfrak{C}_n(1,...,n)\ \mathrm{Pf}'\,\Psi_{n+1},\end{aligned} \tag{3.106}$$

which is easily obtained from the reflectivity property of the Parke-Taylor factor (2.113) and antisymmetry of the insertion operator $S(a,p,a+1) = -S(a+1,p,a)$. From here it is very easy to see that we can obtain an EYM amplitude from

$$\begin{aligned}\sum_{a=1}^{n-1} p\cdot X_a &\left(\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) - (-1)^n\,\mathrm{A}^{(1)}_{n+1}(n,...,a+1,p,a,...,1)\right)\\ &= -\frac{\mathbf{s}^2-\mathbf{c}^2}{\mathbf{s}^2\,\mathbf{c}^2}\int d\mu^{(0)}_{n+1}\, C^h_{pp}\,\mathfrak{C}_n(1,...,n)\ \mathrm{Pf}'\,\Psi_{n+1}\\ &= \frac{\mathbf{s}^2-\mathbf{c}^2}{\mathbf{s}^2\,\mathbf{c}^2}\int d\mu^{(0)}_{n+1}\,\mathfrak{C}_n(1,...,n)\ \mathrm{Pf}(\Psi_p)\ \mathrm{Pf}'\,\Psi_{n+1} = \frac{\mathbf{s}^2-\mathbf{c}^2}{\mathbf{s}^2\,\mathbf{c}^2}\,\mathrm{A}^{\mathrm{EYM}}_{n+1}(1,...,n;p)\ ,\end{aligned}\tag{3.107}$$

where we have used (3.99). All terms $\propto \Lambda$ drop out since the above sum produces a scattering equation $\sum_{a=1}^{n-1} p\cdot X_a\, S(a,p,a+1) = \mathcal{P}_1$ as a prefactor to Λ (2.129). In the last step we have used the definition of an EYM amplitude in the CHY frame (2.119). We can now argue that since EYM amplitudes are unique objects [88], the above sum must be identical to ST-one (2.103) up to a possible multiplicative constant. This concludes the proof of ST-one.

3.10 Factorization Constraints

In this section we want to discuss a very peculiar observation. The analytic form of the linear combination (3.107) is identical to a difference of BCJ relations (2.19) which allows for a replacement of the sub leading collinear limit contributions by full amplitudes, i.e.

$$\begin{aligned}\sum_{a=1}^{n-1} p\cdot X_a &\left(\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) - (-1)^n\,\mathrm{A}^{(1)}_{n+1}(n,...,a+1,p,a,...,1)\right)\\ &= \sum_{a=1}^{n-1}(p_A+p_B)\cdot X_a\left(\mathrm{A}^{p_A||p_B}_{n+1}(1,...,a,p_A,p_B,a+1,...,n)\right.\\ &\qquad\qquad \left. - (-1)^n\,\mathrm{A}^{p_A||p_B}_{n+1}(n,...,a+1,p_A,p_B,a,...,1)\right)\ ,\end{aligned}\tag{3.108}$$

since the leading order collinear limit factorizes into a lower point amplitude (2.84). This formula can be understood as the KK-basis version of ST-one (2.103), i.e. the two indices corresponding to the momenta p_1 and p_n are fixed in their positions. Moreover we can further add the interpretation as a deviation from BCJ relation. Furthermore, it strongly constraints possible factorizations for $\mathrm{A}^{(1)}_{n+1}$. To see this we can assume three possible forms of universal factorization. First, it could be a factorization of the type

$$\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) = \mathrm{Coll}(p_A,p_B,\mathbf{c})\,\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n)\ ,\tag{3.109}$$

i.e. factorization into a lower point amplitude and a hypothetical sub-leading collinear factor $\mathrm{Coll}(p_A,p_B,\mathbf{c})$ which only depends on the collinear momenta and the momentum fractions. Such a behavior would immediately produce a vanishing KK-basis version of ST-one (3.108) since the prefactor can be factored out of the sum and BCJ relations hold for the lower point amplitudes. Note that the same argument is true for the second possibility of a factorization into an amplitude and a prefactor which depends on all momenta. The third possibility is a factorization of the form

$$\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) = \mathrm{Coll}(p_A,p_B,\mathbf{c},p_a,p_{a+1})\,\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n)\ , \quad (3.110)$$

where the prefactor now carries information on the particles adjacent to the collinear legs. If we assume that

$$\mathrm{Coll}(p_A,p_B,\mathbf{c},p_a,p_{a+1}) = -\mathrm{Coll}(p_A,p_B,\mathbf{c},p_{a+1},p_a)\ , \quad (3.111)$$

we would have a vanishing difference

$$\begin{aligned}
&\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) - (-1)^n\,\mathrm{A}^{(1)}_{n+1}(n,...,a+1,p,a,...,1)\\
&= \mathrm{Coll}(p_A,p_B,\mathbf{c},p_a,p_{a+1})\,\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n)\\
&\quad - (-1)^n\,\mathrm{Coll}(p_A,p_B,\mathbf{c},p_{a+1},p_a)\,\mathrm{A}_{n+1}(n,...,a+1,p,a,...,1)\\
&= \left(\mathrm{Coll}(p_A,p_B,\mathbf{c},p_a,p_{a+1}) + \mathrm{Coll}(p_A,p_B,\mathbf{c},p_{a+1},p_a)\right)\,\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n) = 0.
\end{aligned} \quad (3.112)$$

We see that all three types of factorization are inconsistent with the existence of ST-one and thus the existence of EYM amplitudes with one graviton, a mysterious and remarkable observation. However, this does not exclude all types of factorization. In fact, we can combine the above statements together with (3.92) to identify constraints posed on a possible sub-leading collinear theorem. A possible factorization has to be of the form

$$\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) = \mathrm{Coll}(p_A,p_B,\mathbf{c},\{p_{\mathrm{sub}}\})\,\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n)\ , \quad (3.113)$$

where $\{p_{\mathrm{sub}}\}$ denotes particle information of a subset of all particles in the scattering. From (3.92) we must have

$$p\cdot\frac{\partial}{\partial\,\varepsilon_p}\,\mathrm{Coll}(p_A,p_B,\mathbf{c},\{p_a\}) = \frac{\mathbf{c}^2-\mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2}\ . \quad (3.114)$$

From (3.107) we further learn that

$$\begin{aligned}
&\sum_{a=1}^{n-1} p\cdot X_a\left(\mathrm{A}^{(1)}_{n+1}(1,...,a,p,a+1,...,n) - (-1)^n\,\mathrm{A}^{(1)}_{n+1}(n,...,a+1,p,a,...,1)\right)\\
&= \sum_{a=1}^{n-1} p\cdot X_a\Bigg(\mathrm{Coll}(p_A,p_B,\mathbf{c},\{p_{\mathrm{sub}}\})\,\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n)\\
&\qquad\qquad - (-1)^n\,\mathrm{Coll}(p_A,p_B,\mathbf{c},\{\tilde{p}_{\mathrm{sub}}\})\,\mathrm{A}_{n+1}(n,...,a+1,p,a,...,1)\Bigg)\\
&= \sum_{a=1}^{n-1} p\cdot X_a\left(\mathrm{Coll}(p_A,p_B,\mathbf{c},\{p_{\mathrm{sub}}\}) + \mathrm{Coll}(p_A,p_B,\mathbf{c},\{\tilde{p}_{\mathrm{sub}}\})\right)\mathrm{A}_{n+1}(1,...,a,p,a+1,...,n)\\
&\quad = \frac{\mathbf{c}^2-\mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2}\,\mathrm{A}^{\mathrm{EYM}}_{n+1}(1,...,n;p) = \frac{\mathbf{c}^2-\mathbf{s}^2}{\mathbf{c}^2\,\mathbf{s}^2}\sum_{a=1}^{n-1}\varepsilon_p\cdot X_a\,\mathrm{A}^{\mathrm{YM}}_{n+1}(1,...a,p,a+1,n)\ ,
\end{aligned} \quad (3.115)$$

where we used ST-three (2.108) in the last equality and assumed a different subset of momenta $\{\tilde{p}_{\text{sub}}\}$ for the amplitude with reversed arguments. Thus we can read off the second constraint

$$p\, X_a \left(\text{Coll}(p_A, p_B, \mathbf{c}, \{p_{\text{sub}}\}) \; + \text{Coll}(p_A, p_B, \mathbf{c}, \{\tilde{p}_{\text{sub}}\}) \right) = \frac{\mathbf{c}^2 - \mathbf{s}^2}{\mathbf{c}^2\, \mathbf{s}^2}\, \varepsilon_p \cdot X_a \; . \tag{3.116}$$

It remains to see whether those constraints can be solved consistently. A sub-leading collinear theorem cannot be ruled out at this point.

Chapter 4

Universality of Stieberger-Taylor Identities

This chapter is based on the published paper "New relations for graviton-matter amplitudes" [111], written in collaboration with Prof. Jan Plefka.

In this chapter we want to ask whether ST-three (2.108) is universal. In particular, we want to obtain an equivalent formula for scattering amplitudes involving gluons, massive fundamental quarks and massive fundamental scalars. We have studied such amplitudes in section 2.2. Thus the relevant question of this chapter can be formulated as an equation

$$\mathrm{A}^{\mathrm{EsQCD}}(1,2,\ldots,n;p) \stackrel{?}{=} \frac{\kappa}{2g}\sum_{i=2}^{n}(\varepsilon_p \cdot X_i)\,\mathrm{A}^{\mathrm{sQCD}}(1,2,\ldots,i,p,i+1,\ldots)\,, \tag{4.1}$$

where the amplitudes are now in the MJO basis and $X_i = \sum_{a=2}^{i}$.

There are two main motivations for our analysis. The first one is connected to the recently growing interest in EYM amplitudes at tree [45–47] and loop level [42, 51]. In the latter case it is necessary to compute EsQCD tree level amplitudes while using unitarity methods for loop level diagrams. A decomposition into sQCD amplitudes clearly would simplify the intermediate steps due to a general complexity of computations involving gravitons. The second motivation concerns the general understanding of the origin of the double copy construction, e.g. relations like ST-three imply that gravitons are naturally present in YM theory. If the claim of this chapter is true it would extend our understanding by further tying the presence of gravitons to the presence of gluons in scattering processes. This would directly imply that the double copy principle extends beyond the realm of massless amplitudes. Furthermore, it is unclear whether ST-three and the double copy share equivalent principles, i.e. no color-to-kinematics replacement rule is necessary for ST-three. Thus, another goal of this chapter is to find a novel way to understand ST-three from a color-kinematics duality point of view.

In this chapter we will prove universality of ST-three by first giving an first principle argument, for the universality of ST-three, based on graviton and gluon soft limits (2.79), (2.82) and gauge invariance. Next, we will gather evidence from a massive extensions of the CHY formula, presented in [112, 113]. After that we will proof a direct mapping of Feynman graphs with gluons to Feynman graphs where a gluon is replaced by a graviton, using a novel double copy prescription for single color generators. Finally, we will use the MJO color decomposition of sQCD amplitudes (2.38) to directly proof universality for ST-three by comparing it to the MJO decomposition for EsQCD amplitudes while using our novel color-to-kinematics replacement rule.

4.1 Arguments for Universality

Let us investigate ST-three (2.108) from a first principle point of view. As we have explained in sec. 2.5 an amplitude involving gravitons and gluons is entirely fixed from locality and either the correct single soft behavior or gauge invariance for each particle [88]. We may guess the local ansatz

$$\mathrm{A}^{\mathrm{EYM}}_{n,1}(1,...,n;p) \;=\; \sum_{i=2}^{n} \alpha_i\, \mathrm{A}^{\mathrm{YM}}_{n+1}(1,2,...,i,p,i+1,...,n)\ , \tag{4.2}$$

where the graviton is given momentum p and polarization tensor $\varepsilon_p^{\mu\nu} = \varepsilon_p^{\mu}\,\varepsilon_p^{\nu}$. All other particles are gluons and the coefficients α_i have unit mass dimension and must be functions of ε_p in order to ensure bilinearity in the graviton polarization. Note that we have set any proportionality constant to 1 for convenience and identified the indices $n+1=1$. Now we may turn the logic of [88] around and ask whether such an ansatz can be solved for the α_i using first principles. If the answer is yes, the equality is definitely correct. The obvious challenge is that the particle with momentum p is a graviton on the left hand side and a gluon on the right hand side.

Let us implement the single soft limit for particle p on both sides. On the left hand side we will have (2.82)

$$\lim_{p\to 0}\, \mathrm{A}^{\mathrm{EYM}}_{n,1}(1,...,n;p) = \sum_{i=1}^{n} \frac{(\varepsilon_p\cdot p_i)^2}{p\cdot p_i} \mathrm{A}^{\mathrm{YM}}_{n}(1,..,n)\ , \tag{4.3}$$

while on the right hand side we have (2.79)

$$\begin{aligned} &\lim_{p\to 0} \sum_{i=2}^{n} \alpha_i\, \mathrm{A}^{\mathrm{YM}}_{n+1}(1,2,...,i,p,i+1,...,n) \\ &= \sum_{i=2}^{n} \alpha_i \left(\frac{\varepsilon_p\cdot p_i}{p\cdot p_i} - \frac{\varepsilon_p\cdot p_{i+1}}{p\cdot p_{i+1}}\right) \mathrm{A}^{\mathrm{YM}}_{n}(1,2,..,i,i+1,...,n)\ . \end{aligned} \tag{4.4}$$

Our ansatz (4.2) then produces the constraint

$$\sum_{i=1}^{n} \frac{(\varepsilon_p\cdot p_i)^2}{p\cdot p_i} - \sum_{i=2}^{n} \alpha_i \left(\frac{\varepsilon_p\cdot p_i}{p\cdot p_i} - \frac{\varepsilon_p\cdot p_{i+1}}{p\cdot p_{i+1}}\right) = 0\ . \tag{4.5}$$

We can solve this ansatz for the α_i by collecting terms for every pole $\frac{\varepsilon_p\cdot p_i}{p\cdot p_i}$ for every i. This results in a system of linear equations

$$\begin{aligned}
i = 1: &\qquad \varepsilon_p \cdot p_1 + \alpha_n = 0 \ , \\
i = 2: &\qquad \varepsilon_p \cdot p_2 - \alpha_2 = 0 \ , \\
i = 3: &\qquad \varepsilon_p \cdot p_3 - \alpha_3 + \alpha_2 = 0 \ , \\
&\vdots \\
i = k: &\qquad \varepsilon_p \cdot p_k - \alpha_k + \alpha_{k-1} = 0 \ , \\
&\vdots \\
i = n: &\qquad \varepsilon_p \cdot p_n - \alpha_n + \alpha_{n-1} = 0 \ .
\end{aligned} \tag{4.6}$$

The unique solution is given by

$$\alpha_i = \varepsilon_p \cdot \left(\sum_{a=2}^{i} p_a \right) = \varepsilon_p \cdot X_i \ , \tag{4.7}$$

where we again use the notion of a region momentum X_i. Note that this solution is only valid on the support of momentum conservation and on-shell conditions. Plugging it back into (4.2) we arrive at

$$\mathrm{A}^{\mathrm{EYM}}_{n,1}(1, ..., n; p) \ = \sum_{i=2}^{n} \varepsilon_p \cdot X_i \, \mathrm{A}^{\mathrm{YM}}_{n+1}(1, ..., i, p, i+1, ..., n) \ , \tag{4.8}$$

which is identical to ST-three (2.108). From the point of view of our derivation we still have to check that both sides have the correct soft behavior in the rest of the particles. It is instructive to see the correct behavior for a low point example which trivially generalizes to arbitrary points. Consider the leading part of the gluon soft limit $p_2 \to 0$ for a six particle scattering. On the left hand side of (4.8) we get the universal behavior

$$\left(\frac{\varepsilon_2 \cdot p_1}{p_2 \cdot p_1} - \frac{\varepsilon_2 \cdot p_3}{p_2 \cdot p_3} \right) \mathrm{A}^{\mathrm{EYM}}(1, 3, 4, 5; p) \ , \tag{4.9}$$

and on the right hand side we take the limit of

$$\begin{aligned}
&\varepsilon_p \cdot (p_2 + p_3) \ \mathrm{A}^{\mathrm{YM}}(1, 2, 3, p, 4, 5) \\
&+ \varepsilon_p \cdot (p_2 + p_3 + p_4) \ \mathrm{A}^{\mathrm{YM}}(1, 2, 3, 4, p, 5) + \varepsilon_p \cdot (p_2 + p_3 + p_4 + p_5) \ \mathrm{A}^{\mathrm{YM}}(1, 2, 3, 4, 5, p) \ ,
\end{aligned} \tag{4.10}$$

which is

$$\begin{aligned}
&\left(\frac{\varepsilon_2 \cdot p_1}{p_2 \cdot p_1} - \frac{\varepsilon_2 \cdot p_3}{p_2 \cdot p_3} \right) \Big(\varepsilon_p \cdot p_3 \mathrm{A}^{\mathrm{YM}}(1, 3, p, 4, 5) + \varepsilon_p \cdot (p_3 + p_4) \ \mathrm{A}^{\mathrm{YM}}(1, 3, 4, p, 5) \\
&+ \varepsilon_p \cdot (p_3 + p_4 + p_5) \, \mathrm{A}^{\mathrm{YM}}(1, 3, 4, 5, p) \Big) = \left(\frac{\varepsilon_2 \cdot p_1}{p_2 \cdot p_1} - \frac{\varepsilon_2 \cdot p_3}{p_2 \cdot p_3} \right) \mathrm{A}^{\mathrm{EYM}}(1, 3, 4, 5; p),
\end{aligned} \tag{4.11}$$

thus yielding the correct soft behavior in gluon 2. Obviously this generalizes to any gluon index and any multiplicity. We therefore conclude that (4.8) has the correct soft behavior in all particles. Gauge invariance is also trivially realized for all gluons due to the choice of expanding the EYM amplitude in terms of YM amplitudes. Only for the graviton we encounter a non-triviality. The graviton is bilinear in its polarization. On the left hand side of (4.8) gauge invariance is trivially manifest. On the right hand side one half of the polarization is encoded in a YM amplitude and the other half in the prefactor of the YM amplitudes. Shifting the polarization in the YM amplitudes is trivially gauge invariant. A shift in the prefactor on the other hand enforces the condition

$$0=\sum_{i=2}^{n} p\cdot X_i\,\mathrm{A}^{\mathrm{YM}}_{n+1}(1,...,i,p,i+1,...,n)\ , \tag{4.12}$$

which are exactly the fundamental BCJ relations (2.19). This result allows for a remarkable interpretation of BCJ relations. They can be viewed as a necessary condition for EYM amplitudes to be gauge invariant.

We see that the leading soft limit and gauge invariance is manifest for each particle in (4.8) thus uniquely fixing the EYM amplitude through a linear combination of YM amplitudes according to [88]. It is crucial to note that the only ingredients used are single gluon/graviton soft limits and BCJ relations. This allows us to make the following intuitive guess. Since leading soft limits are universal and we also noted that BCJ relations are universal (2.47), we can assume that ST-three has to be universal aswell. Of course, our previous derivation is based on the proven fact that EYM amplitudes are uniquely fixed by first principles, which is not proven, and most likely not true, for EsQCD amplitudes. Still, it serves as a guiding principle and yields enough indication and motivation to pursue a deeper technical analysis.

4.2 Evidence for Universality

In this section we shall turn to the first technical evidence that ST-three is universal based on the CHY formula. We first start by giving the most elegant proof of ST-three (2.108) based on the CHY formula for EYM amplitudes (2.119). A color ordered amplitude for an EYM amplitude with one graviton reads

$$\mathrm{A}^{\mathrm{EYM}}(1,...,n;p)=\int_{\mathbb{CP}^1} d\mu_{n,p}\,\mathfrak{C}(1,...,n)\ \mathrm{Pf}\,\Psi_p\ \mathrm{Pf}'\,\Psi_{n,p}\ , \tag{4.13}$$

where in our notation we separate the gluon and graviton indices by a comma in the CHY building blocks. The matrix Ψ_p is given by

$$\begin{pmatrix} 0 & -C_{pp} \\ C_{pp} & 0 \end{pmatrix}\ , \tag{4.14}$$

with the Pfaffian (3.47)

$$\mathrm{Pf}\,\Psi_p=-C_{pp}=\sum_{a=1}^{n}\frac{2\,\varepsilon_p\cdot p_a}{\sigma_{pa}}\ . \tag{4.15}$$

We rewrite C_{pp} using insertion operators (2.121) as

$$C_{pp} = -2 \sum_{a=2}^{n} \varepsilon_p \cdot X_a \, S(a,p,a+1) \ , \tag{4.16}$$

which can easily be proven using on-shell conditions and momentum conservation similar to (2.130). Using (2.123) we immediately arrive at ST-three

$$\sum_{\sqcup\!\sqcup} \varepsilon_p \cdot X_p \, \mathrm{A}^{\mathrm{YM}}(1,2,\{3,...,n\} \sqcup\!\sqcup \{p\}) = \mathrm{A}^{\mathrm{EYM}}(1,...,n;p) \ , \tag{4.17}$$

where we ignore any proportionality constant for convenience.

Recently, massive extensions of the CHY formula have been studied by S.G. Naculich [112, 113]. In his works it is shown that it is possible to include up to three massive particles, with spin $s \leq 1$, into the CHY formula without any change to its analytic structure. This result immediately proves ST-three in the presence of up to three massive particles as shown above. Of course this is not a prove of generic universality but we may wonder whether there is anything special about three massive particles. In Naculich's work, one may only include three massive particles in the CHY formula due to modular gauge fixing which allows three punctures to be given fixed numeric values. This results in an elimination of the mass properties of three particles from final results. But from and amplitudes point of view no conformal symmetry needs to be fixed and thus there is no reason that ST-three shall only work with up to three massive particles (or at all). This is very compelling evidence for a generic universality of ST-three.

4.3 Proving Universality

Now that we have gathered a convincing amount of arguments and evidence that ST-three is universal, we aim to prove it.

4.3.1 Strategy

We will apply the following strategy.

As a first step we will compute two full amplitudes. The first one will involve n particles from sQCD theory with at least one external gluon which we will treat distinctively, i.e. giving it momentum p and polarization ε_p. The other amplitude will involve the exact same particle content except that the distinct gluon is replaced by a graviton with momentum p and polarization tensor $\varepsilon_p \, \varepsilon_p$.

The second step is to compare both amplitudes while trying to identify a simple color-to-kinematics replacement rule which maps the sQCD amplitude into the EsQCD one.

Step three is then to use the MJO decompositions (2.38),(2.77) for both amplitudes in which we apply the, newly found, replacement rule such that we can identify a universal version of ST-three.

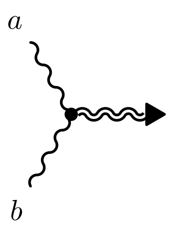
The obvious challenge regarding this strategy is the computation of both amplitudes. One way to make out lives easier is to perform a combinatoric trick regarding the sum of all Feynman graphs. If we want to compute an n point amplitude with one external gluon through the sum of all Feynman graphs, we can obtain the same result by summing all possible insertions of an on-shell gluon to every Feynman graph in the $n-1$ point amplitude without the gluon [1]. This is obviously true and also valid if we replace the gluon by a graviton. Since all possible insertions are dictated by Feynman rules involving gluons/gravitons, we only need to understand how the Feynman rules of sQCD and EsQCD behave if we take one gluon/graviton on-shell.

Note that we are aiming at a relation at order $\mathcal{O}(\kappa)$ which implies that we do not consider any graphs involving internal gravitons.

4.3.2 Feynman Rules with one External on-shell Gluon/Graviton

We have already established all Feynman rules in sQCD and EsQCD up to one external graviton in sec. 2.4. We construct on-shell Feynman rules by contracting one external graviton or gluon with its polarization tensor/vector and using the conditions $\varepsilon_p \cdot p = \varepsilon_p^2 = 0$. Graphically we denote the on-shell leg with a triangle. We begin our analysis with on-shell graviton insertions.

The first on-shell Feynman rule is the two-gluon-one-graviton-vertex. It can represent an insertion of a graviton to an external or a propagating gluon in any Feynman diagram. The expression is given by



$$= i\kappa g\, \delta^{ab} \left[\frac{1}{2}(p_a^2 + p_b^2)\varepsilon_p^{\mu_a}\varepsilon_p^{\mu_b} + (\varepsilon_p \cdot p_a)(M^{\mu_a\mu_b} + W^{\mu_a\mu_b}) \right] \;, \tag{4.18}$$

where we defined

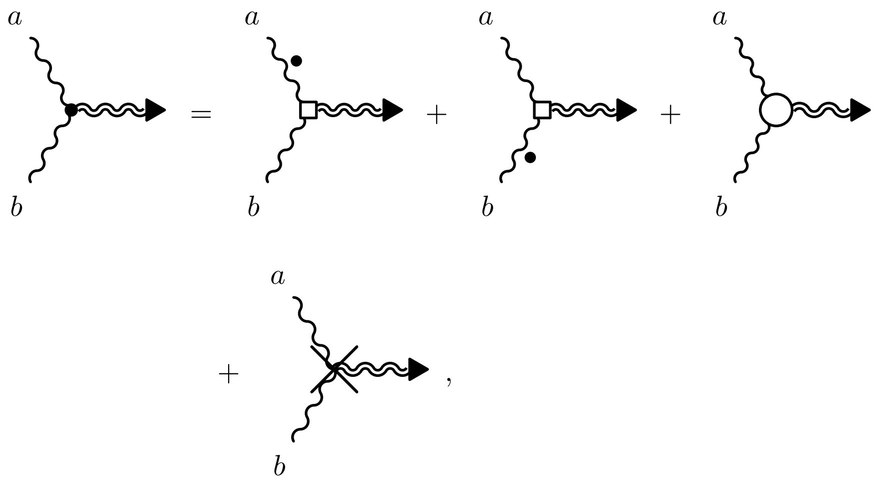
$$\begin{aligned} M^{\mu\nu} &= (\varepsilon_p \cdot p_a)\, \eta^{\mu_a\mu_b} + \varepsilon_p^{\mu_a}\, p^{\mu_b} - \varepsilon_p^{\mu_b}\, p^{\mu_a} \\ W^{\mu\nu} &= \varepsilon_p^{\mu_a}\, p_b^{\mu_b} - \varepsilon_p^{\mu_b}\, p_a^{\mu_a} \;, \end{aligned} \tag{4.19}$$

and used the on-shell property of the graviton

$$p_a \cdot p_b = \frac{1}{2}\left((p_a + p_b)^2 - p_a^2 - p_b^2 \right) \;, \tag{4.20}$$

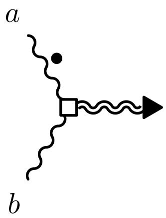
where the first term in the bracket is $p^2 = 0$ by momentum conservation. Notationally we decompose this expression diagrammatically through new Feynman rules

[1]This approach is sometime referred to in the literature [72] as the radiation vertex expansion.

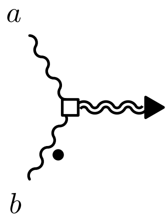


$$(4.21)$$

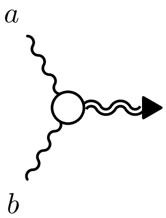
And identify



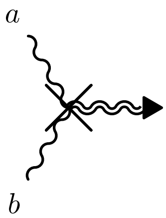
$$= \frac{i\kappa g}{2} p_a^2 \varepsilon_p^{\mu_a} \varepsilon_p^{\mu_b} \delta^{ab} \ , \tag{4.22}$$



$$= \frac{i\kappa g}{2} p_b^2 \varepsilon_p^{\mu_a} \varepsilon_p^{\mu_b} \delta^{ab} \ , \tag{4.23}$$



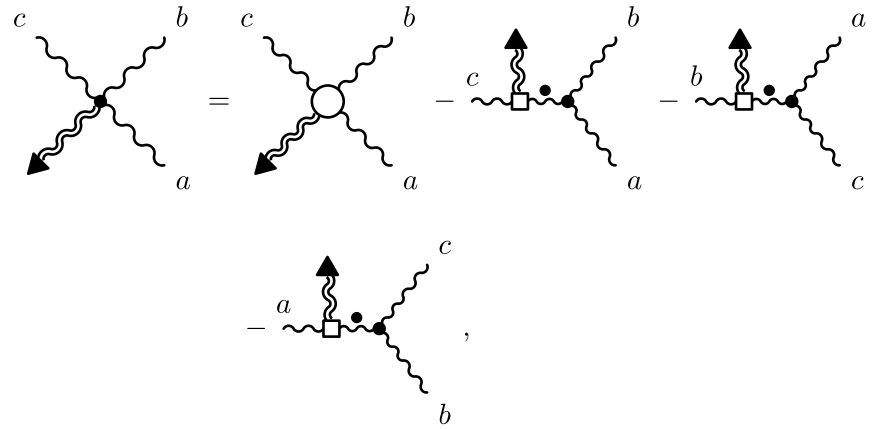
$$= i\kappa g \left(\varepsilon_p \cdot p_a\right) M^{\mu_a \mu_b} \delta^{ab} \ , \tag{4.24}$$



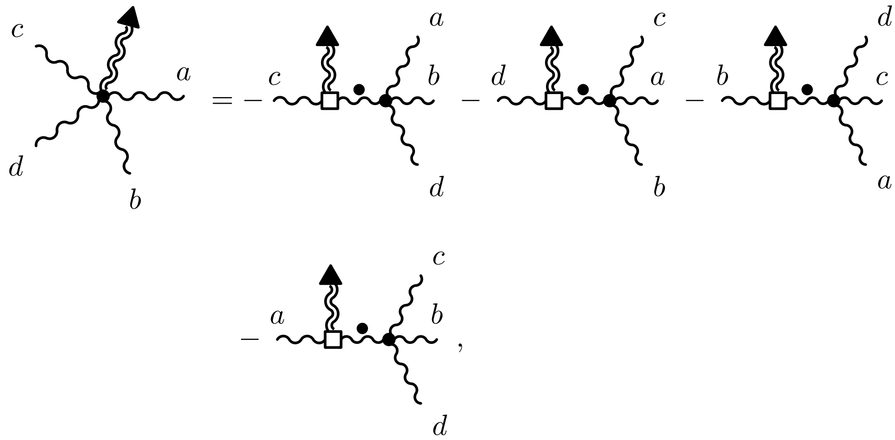
$$= i\kappa g \left(\varepsilon_p \cdot p_a\right) W^{\mu_a \mu_b} \delta^{ab} \ . \tag{4.25}$$

The dot on top of a gluon line denotes the appearance of the inverse massless propagator, p_a^2 or p_b^2, attached to the corresponding leg. Note that in any tree level computation of a scattering amplitude the term $W^{\mu_a\mu_b}(p_a, p_b)$ will vanish due to on-shell Ward identities (2.27), a similar observation has been presented in [72]. The graph with the big white bubble in the middle (4.24) is our first effective Feynman rule which will play an important role shortly.

Next, we study the insertion of a graviton directly into the YM vertices, producing a three-gluon-one-graviton-vertex or a four-gluon-one-graviton vertex. In both cases a remarkable decomposition can be found involving the boxed Feynman rules (4.22), (4.23). In particular, the expressions reads

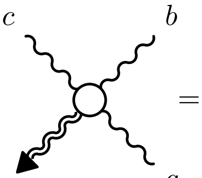


(4.26)



(4.27)

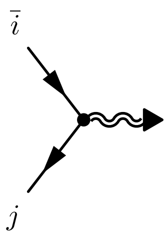
with the identification of our second effective Feynman rule



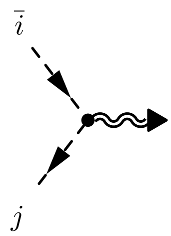
$$= \frac{\kappa g}{2} f^{abc} \Big[\eta^{\mu_a \mu_b} \varepsilon_p^{\mu_c} (\varepsilon_p \cdot (p_b - p_a)) + \eta^{\mu_a \mu_c} \varepsilon_p^{\mu_b} (\varepsilon_p \cdot (p_a - p_c)) + \eta^{\mu_b \mu_c} \varepsilon_p^{\mu_a} (\varepsilon_p \cdot (p_c - p_b)) \Big] \, . \tag{4.28}$$

Clearly, we found a very powerful decomposition since in the sum of all Feynman graphs we will add all graviton insertions into vertices to graviton insertions into the external legs of the vertices. This results in a massive cancellation of all boxed Feynman rules in all graphs. An immediate, beautiful, result is that a graviton insertion into the four gluon vertex will completely cancel with graviton insertions to all legs of the same vertex and similarly only the effective Feynman rules (4.28), (4.24) contribute to any amplitude calculation in pure YM.

We now turn to graviton insertions into massive fundamental matter, i.e. Feynman rules including either quarks or scalars. We begin with insertions into propagators/external legs

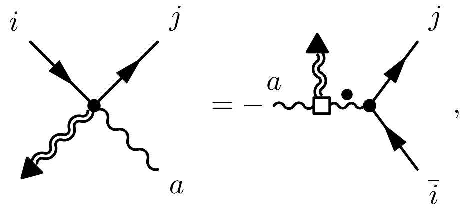


$$= -\frac{i\kappa}{2}\,\not{\varepsilon}_p\,(\varepsilon_p \cdot p_i)\delta^{j\bar{i}} \,, \quad (4.29)$$

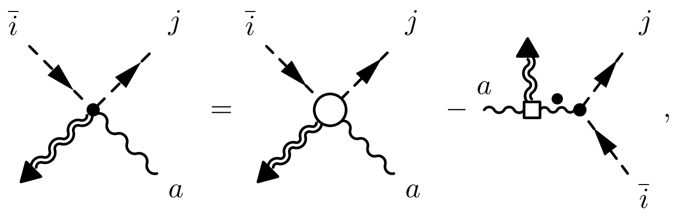


$$= i\kappa\,(\varepsilon_p \cdot p_i)\,(\varepsilon_p \cdot p_j)\,\delta^{j\bar{i}} \,, \tag{4.30}$$

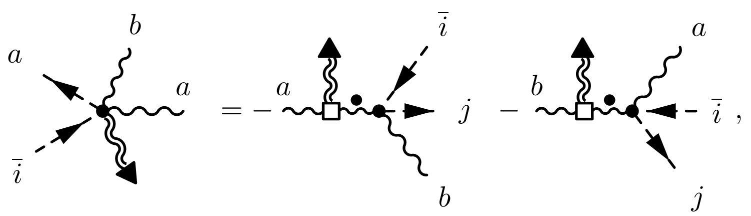
followed by vertex insertions. They are again subject to a decomposition into effective and boxed Feynman rules which cancel with external gluon leg corrections.



(4.31)

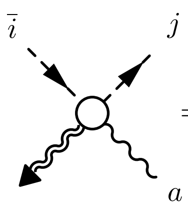


(4.32)



(4.33)

where we encounter another effective Feynman rule

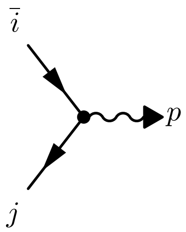


$$= -\frac{i\kappa g}{2}\,\varepsilon_p^{\mu_a}(\varepsilon_p \cdot (p_i - p_j))\,(T^a)_{j\bar{i}} \,. \tag{4.34}$$

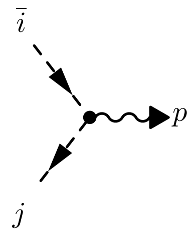
Having established the form of all graviton insertions into an arbitrary tree level amplitude, we may summarize our findings in a remarkable rule.

Rule 1: A tree level amplitude with one external graviton and an arbitrary number of gluons and massive fundamental matter (scalars and quarks) is computed from the sole use of effective Feynman rules (4.28), (4.34), (4.24) and quarks/scalar line corrections (4.29), (4.30). Therefore, the number of graviton insertions into a given sQCD tree level amplitude is equivalent to the number of gluon insertion into the same amplitude.

We now turn to the insertion of an on-shell gluon to a given sum of Feynman diagrams. A gluon can be added to a quark, scalar and gluon propagator turning them into three-vertices. We start with the insertion into a quark or scalar propagator. We have



$$= ig\, \not{\varepsilon}_p\, (T^p)_{j\bar{i}} \tag{4.35}$$



$$= 2ig\, (\varepsilon_p \cdot p_i)\, (T^p)_{j\bar{i}} \tag{4.36}$$

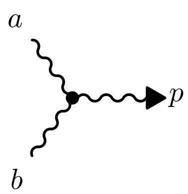
where we used momentum conservation in the two-scalar-one-gluon Feynman rule $p_j = -p_i - p_a$.

At this point it is worthwhile to take a close look at a remarkably simple color-to-kinematics replacement rule, i.e. we can obtain the equivalent vertices but with an external graviton instead of the gluon (4.30), (4.29) via the prescription

$$(T^p)_{j\bar{i}} \to \frac{\kappa}{2g}(\varepsilon_p\, p_j)\, \delta^{j\bar{i}} \quad , \quad (T^p)_{j\bar{i}} \to -\frac{\kappa}{2g}(\varepsilon_p\, p_i)\, \delta^{j\bar{i}} \,, \tag{4.37}$$

where $\delta^{j\bar{i}}$ is the Kronecker delta in color space. More specifically, we replace the fundamental SU(N) generator, associated to the gluon color and carrying fundamental/anti-fundamental indices belonging to the quark/scalar colors, with the product of the gluon polarization vector times the momentum associated with the leg that carries either the fundamental (plus sign) or anti-fundamental (minus sign) color of the quark/scalar. Note that both replacement rules are equal on the support of momentum conservation and on-shell conditions.

Further investigation reveals a similar structure if the gluon is inserted into a gluon propagator. We straightforwardly compute



$$= 2g f^{abp}\left[M^{\mu_a\mu_b} + \frac{1}{2}\, W^{\mu_a\mu_b}\right] \,, \tag{4.38}$$

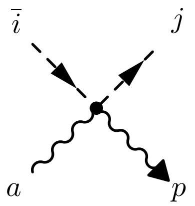
where we use the same notation as in (4.19). Ignoring the Ward identity term, we can generate the effective vertex with one external graviton (4.24) through

$$f^{apb} = i(T^p)_{ab} \to -i\frac{\kappa}{2g}(\varepsilon_p\, p_a)\, \delta^{ab} \,. \tag{4.39}$$

Note the difference in sequence of the color indices between the Feynman rule and the replacement rule. We observe that the prescription of the fundamental color factors differs to the adjoint factors by a minus sign. Nevertheless, we can define adjoint color factors

$$\tilde{f}^{abc} = i f^{abc} \ , \tag{4.40}$$

which will have the exact same behavior in the color to kinematics prescription. Higher multiplicity vertices can also be generated by this prescription. We start with the case

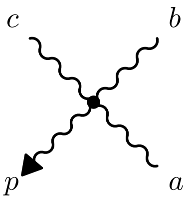


$$= ig^2 \varepsilon_p^{\mu_a} \left((T^a)_{jk} (T^p)_{k\bar{i}} + (T^p)_{jk} (T^a)_{k\bar{i}} \right) \tag{4.41}$$

Sending

$$(T^p)_{k\bar{i}} \to = -\frac{\kappa}{2g} (\varepsilon_p \cdot p_i)\, \delta^{k\bar{i}} \quad , \quad (T^p)_{jk} \to \frac{\kappa}{2g} (\varepsilon_p \cdot p_j)\, \delta^{jk} \ , \tag{4.42}$$

we can see that we land exactly on (4.34). The four gluon vertex is more involved. The expression reads



$$\begin{aligned} = -ig^2 \Big[& \varepsilon_p^{\mu_c}\, \eta^{\mu_b \mu_a} (f^{cbe} f^{pae} - f^{cae} f^{bpe}) - \eta^{\mu_c \mu_a}\, \varepsilon_p^{\mu_b} \, (f^{cbe} f^{pae} + f^{cpe} f^{bae}) \\ & + \eta^{\mu_c \mu_b}\, \varepsilon_p^{\mu_a} (f^{cpe} f^{bae} + f^{cae} f^{bpe}) \Big] \ . \end{aligned} \tag{4.43}$$

Let us focus on the first color structure

$$\begin{aligned} f^{cbe} f^{pae} - f^{cae} f^{bpe} &= -(f^{cbe} f^{ape} + f^{cae} f^{bpe}) = -i (f^{cbe}\, (T^p)_{ae} + f^{cae} (T^p)_{be}) \\ & \to i \frac{\kappa}{2g} \Big(f^{cbe}\, (\varepsilon_p \cdot p_a)\, \delta^{ae} + f^{cae} (\varepsilon_p \cdot p_b) \Big)\, \delta^{be} = i \frac{\kappa}{2g} \Big(f^{cba}\, (\varepsilon_p \cdot p_a) + f^{cab} (\varepsilon_p \cdot p_b) \Big) \\ & = i \frac{\kappa}{2g} f^{abc}\, (\varepsilon_p \cdot (p_b - p_a)) \ . \end{aligned} \tag{4.44}$$

Hence

$$-ig^2 \Big[\varepsilon_p^{\mu_c}\, \eta^{\mu_b \mu_a} (f^{cbe} f^{pae} - f^{cae} f^{bpe}) \Big] \to \frac{g\kappa}{2}\, \varepsilon_p^{\mu_c}\, \eta^{\mu_b \mu_a}\, f^{abc}\, (\varepsilon_p \cdot (p_b - p_a)) \ , \tag{4.45}$$

which is exactly the first term in (4.28). The other two terms can be seen to map correctly in the exact same way.

We may again summarize the above observations into a second rule.

Rule 2: The on-shell Feynman rules need for the computation of a tree level amplitude with gluons, fundamental matter (quarks/scalars) and one external graviton (4.28), (4.34), (4.30), (4.29), and therefore full diagrams including such vertices, can be obtained through the color-to-kinematics replacement rule

$$\mathcal{R}_p := \{(T^p)_{j\bar{i}} \to \frac{\kappa}{2\,g}(\varepsilon_p \cdot k_j)\,\delta_{j\bar{i}}\,,\ \tilde{f}^{apb} \to \frac{\kappa}{2g}(\varepsilon_p \cdot k_a)\,\delta_{ab}\}\ , \tag{4.46}$$

from the same Feynman rules where the on-shell graviton is replaced by an on-shell gluon.

We end this section by, straightforwardly, combining both rules 4.3.2, 4.3.2 into the corollary

$$\mathcal{A}^{\text{tree}}_{n,k;1} = \mathcal{A}^{\text{tree}}_{n+1,k,0}\Big|_{R_p}\ , \tag{4.47}$$

where we have k fundamental/anti-fundamental pairs, $n-2k$ gluons and one graviton on the left hand side. This gravitons is an "upgraded" gluon on the right hand side. This is the desired color-to-kinematics replacement rule which we wanted to achieve such that ST-three originates from a color-to-kinematics replacement rule similar to the double copy prescription.

4.3.3 Proof

We are now ready to prove universality of ST-three (2.108). As we have already mentioned, we will do this by writing

$$\mathcal{A}^{\text{tree}}_{n,k;1} = \mathcal{A}^{\text{tree}}_{n+1,k,0}\Big|_{R_p}\ , \tag{4.48}$$

in its MJO-color basis and read off the result. The expressions read

$$\mathcal{A}^{\text{tree}}_{n,k;1} = \sum_{\sigma\in\text{MJO basis}}^{\chi(n,k)} C(\underline{1},\bar{2},\sigma)\,\text{A}^{\text{EsQCD}}(\underline{1},\bar{2},\sigma)\ , \tag{4.49}$$

$$\mathcal{A}^{\text{tree}}_{n+1,k;0} = \sum_{\sigma\in\text{MJO basis}}^{\chi(n+1,k)} C(\underline{1},\bar{2},\sigma)\,\text{A}^{\text{sQCD}}(\underline{1},\bar{2},\sigma)\Big|_{R_p}\ , \tag{4.50}$$

The color factor of the $\mathcal{A}^{\text{tree}}_{n+1,k;0}$ can be easily rewritten such that the decomposition reads

$$\begin{aligned}\mathcal{A}^{\text{tree}}_{n+1,k;0} &= \sum_{\sigma\in\text{MJO basis}}^{\chi(n,k)} \sum_{\shuffle} C(\underline{1},\bar{2},\{\sigma\}\shuffle\{p\})\Big|_{R_p}\,\text{A}^{\text{sQCD}}(\underline{1},\bar{2},\{\sigma\}\shuffle\{p\})\\ &= \sum_{\sigma\in\text{MJO basis}}^{\chi(n,k)} \sum_{i=0}^{|\sigma|} C(\underline{1},\bar{2},...,\sigma_i,p,\sigma_{i+1},...)\Big|_{R_p}\,\text{A}^{\text{sQCD}}(\underline{1},\bar{2},...,\sigma_i,p,\sigma_{i+1},...)\ ,\end{aligned} \tag{4.51}$$

where we identify $\sigma_0 = \bar{2}$ and $\sigma_{|\sigma|+1} = \sigma_0$. The above expression reveals that (4.1) is proven once we prove

$$C(\underline{1},\bar{2},...,\sigma_i,p,\sigma_{i+1},...)\Big|_{R_p} = \frac{\kappa}{2g}(\varepsilon_p \cdot k_{2\sigma})\,C(\underline{1},\bar{2},...,\sigma_i,\sigma_{i+1},...) \tag{4.52}$$

where $k_{2\sigma_i} := k_2 + k_{\sigma_1} + \dots k_{\sigma_i}$.

We start with a prove for $i = 0$. From (2.42) we have

$$C(\underline{1}, \bar{2}, p, \sigma) = \{2|\bar{T}^{a_p}\bar{\Omega}|1\}\dots = (-)^{\Omega+1}\{1|\Omega\, T^{a_p}|2\}\dots \tag{4.53}$$

where Ω is determined by the Dyck word of σ. With $T_{a_p}|2\}|_{\mathcal{R}_{a_p}} \to -|2\}\,\frac{\kappa}{2g}(\varepsilon_p \cdot k_2)$ we easily find (4.52). We proceed with the non-trivial proofs. Three cases have to be considered separately.

Case $\boldsymbol{\sigma_i}$ is a gluon: In this case we proceed inductively. We define $\sigma_i = c$ and use the short-hand notation

$$C(\underline{1}, \bar{2}, \sigma_1 \dots \sigma_i, p, \sigma_{b+1} \dots \sigma_{n-1}) =: C_{\dots|cp|\dots}\,,$$

while we also assume that (4.52) is true for $C_{\dots|pc|\dots}$. We now zoom in into the analytic structure of $C_{\dots|cp|\dots}$ at the position of p. We write this as

$$C_{\dots|cp|\dots} = \dots \Xi^c_l\, \Xi^p_l \dots\ . \tag{4.54}$$

Note that the two gluons are next to each other and thus have the same level of nestedness. We now can compute the commutator

$$C_{\dots|cp|\dots} - C_{\dots|pc|\dots} = \dots [\Xi^c_l, \Xi^p_l] \dots = \dots \tilde{f}^{cpd}\Xi^d_l \dots\ , \tag{4.55}$$

where we have used the Lie-algebra structure of Ξ (2.44). At this point it is easy to see that

$$C_{\dots|cp|\dots} - C_{\dots|pc|\dots}\Big|_{\mathcal{R}_{a_p}} = \frac{\kappa}{2g}(\varepsilon_p \cdot k_c)\, C_{\dots|c|\dots}\,. \tag{4.56}$$

Thus, the induction step is performed due to our assumption that

$$C_{\beta|pc|\dots}|_{\mathcal{R}_{a_p}} = (\varepsilon_p \cdot k_{2\,\sigma})\, C_{\beta|c|\dots}\ , \tag{4.57}$$

with $\beta = \{\sigma_1 \dots \sigma_i\}$.

Case $\boldsymbol{\sigma_i}$ is a fundamental: Again, we proceed inductively. We zoom in at the position of p

$$\begin{aligned} C_{\dots|cp|\dots} &= \cdots |c\}\Xi^p_l \cdots\ , \\ C_{\dots|pc|\dots} &= \cdots \Xi^p_{l+1}|c\} \cdots = \cdots T^p|c\} \cdots + \cdots |c\}\Xi^p_l \cdots\ . \end{aligned} \tag{4.58}$$

The commutator is then given by

$$C_{\dots|cp|\dots} - C_{\dots|pc|\dots} = -\cdots T^p|c\} \cdots\ , \tag{4.59}$$

and we immediately obtain

$$\big(C_{\dots|cp|\dots} - C_{\dots|pc|\dots}\big)\big|_{\mathcal{R}_p} = (\varepsilon_p \cdot k_c)\, C_{\dots|c|\dots}\ . \tag{4.60}$$

This concludes the induction step.

Case $\boldsymbol{\sigma_i}$ is an anti-fundamental: We zoom in on the position of p

$$C_{\dots|cp|\dots} = \cdots \{c|\, T^c \otimes \Xi^c_{l-1}\, \Xi^p_l \cdots = \cdots \{c|\, (T^c\, T^p) \otimes \Xi^c_{l-1} \cdots + \cdots \{c|\, (T^c \otimes (\Xi^c_{l-1}\, \Xi^p_{l-1}) \cdots \; , \tag{4.61}$$

where we used the definition (2.43). Similarly we get

$$C_{\dots|pc|\dots} = \cdots \Xi^p_{l-1}\, \{c|\, T^c \otimes \Xi^c_{l-1} \cdots = \cdots \{c|\, (T^c \otimes (\Xi^p_{l-1}\, \Xi^c_{l-1}) \cdots \; . \tag{4.62}$$

Now computing the commutator will give

$$\begin{aligned} C_{\dots|cp|\dots} - C_{\dots|pc|\dots} &= \cdots \{c|\, (T^c\, T^p) \otimes \Xi^c_{l-1} \cdots + \cdots \{c|\, T^c \otimes \left([\Xi^c_{l-1}\, , \,\Xi^p_{l-1}]\right)\right) \cdots \\ &= \cdots \{c|\, (T^p\, T^c) \otimes \Xi^c_{l-1} \cdots + \cdots \{c|\, ([T^c\, , \, T^p]) \otimes \Xi^c_{l-1} \cdots + \cdots \{c|\, T^c \otimes \left([\Xi^c_{l-1}\, , \,\Xi^p_{l-1}]\right)\right) \cdots \\ &= \cdots \{c|\, (T^p\, T^c) \otimes \Xi^c_{l-1} \cdots + \cdots \tilde{f}^{cpk}\{c|\, \left(T^k\right) \otimes \Xi^c_{l-1} \cdots + \cdots \tilde{f}^{cpk}\{c|\, (T^c) \otimes \Xi^k_{l-1} \cdots \\ &= \cdots \{c|\, (T^p\, T^c) \otimes \Xi^c_{l-1} \cdots \; , \end{aligned} \tag{4.63}$$

immediately implying

$$\left(C_{\dots|cp|\dots} - C_{\dots|pc|\dots}\, \right)\big|_{\mathcal{R}_p} = (\varepsilon_p \cdot k_c)\, C_{\dots|c|\dots} \; . \tag{4.64}$$

This again completes the induction step.

We therefore conclude the proof of (4.52) and thus we complete the proof of (4.1) relating single graviton E)QCD amplitudes to sQCD ones through a universal version of ST-three.

4.4 Singe particle color-kinematics duality

One crucial ingredient in the double copy procedure is the existence of color dual numerators, i.e. kinematics which satisfy Lie-algebra Jacobi relations (2.100). We can check whether our replacement rule (4.46) also satisfies Lie-algebra relations. Indeed, we can easily obtain the relation

$$([\Xi^c_l, \Xi^p_l])_{i\bar{j}}\Big|_{\mathcal{R}_{a_p}} = \frac{\kappa}{2g}(\varepsilon_p \cdot k_c)\, (\Xi^c_l)_{i\bar{j}}\, , \tag{4.65}$$

from the use of our replacement rule in (2.44). Thus we conclude, that we have found a single particle form of color-kinematics duality which tightens our replacement rule to double copy principles.

Chapter 5

Double Copy between Classical Effective Actions

This chapter is based on the published paper "Effective action of dilaton gravity as the classical double copy of Yang-Mills theory" [114], written in collaboration with Prof. Jan Plefka and Dr. Jan Steinhoff.

In this chapter we aim at an extension of a curios set of recent observations regarding applications of the double copy 2.6 outside the context of scattering amplitudes.

Recently, the double copy has been observed in classical systems, i.e. non-abelian classical worldlines are related to gravitating worldlines through a novel double copy prescription [31–33, 115]. In particular, classical, perturbative solutions to the equations of motion of gluon radiation is mapped to solutions of graviton radiation. Such classical systems may be seen as a black hole binary, which makes such a double copy potentially relevant for the prediction of gravitational wave signals from such systems. Another direction of the classical double copy was considered in [116]. There the double copy is performed for amplitudes with external, massive fundamental scalars interacting in YM, on the gauge theory side, and in dilaton-gravity, on the gravity side. Afterwards the classical limit of the resulting gravity amplitude is taken, reproducing the results obtained from the classical double copy.

Here, we want to take a different approach. We aim at a double copy between classical effective actions of two classical worldlines interacting in gauge theory and gravity. In contrast to the direction of double copies between solutions to equation of motion, which start and end at a classical level, we are ascending from a QFT context into the classical regime similar to [116]. Our motivation is mainly rooted in the goal of getting further insides into the inner workings of the double copy, i.e. discovering its limits.

A classical limit separates the non-quantum from the quantum by the presence (or absence) of Planck's constant $\hbar$. Therefore, it will be advantageous to reintroduce $\hbar$ in all quantities, i.e. going back from natural units to SI units. Furthermore, we will work exclusively in position space due to convenience and curiosity, since QFT calculations are usually performed in momentum space and it would be nice to observe a double copy in position space.

This chapter is organized as follows. In the first section we will construct two systems which we want to map through the double copy. In both systems we will have two classical spinless massive worldlines which will be color charged and interacting in YM on the gauge theory side and charged under a dilaton force while interacting through dilaton-gravity on the gravity side. In the second section we will compute the associated effective actions. Finally, in the third section, we will propose a double copy prescription whose validity we are going to proof.

5.1 Setting the stage

Our goal is to observe a double copy structure between the classical effective actions of two systems consisting of two spinless, classical worldlines of point particles which interact through YM, on the one hand, and dilaton-gravity, on the other hand. In this section we are going to construct both systems and give their corresponding full quantum actions. We start with the dilaton-gravity case. Note that we do not need an axion field in our computation. A coupling of a worldline to an axion is connected to non vanishing spin.

5.1.1 Worldlines interacting through dilaton-gravity

The action of dilaton-gravity (dg) is defined as[1]

$$S_{\text{dg}} = S_{\text{EH}} + S_{\text{Dilaton}} \tag{5.1}$$

$$= -\frac{2}{\kappa^2}\int d^4x\sqrt{-g}\left[R - 2\partial_\mu\phi\partial^\mu\phi\right] + (\text{GHY boundary term}) \tag{5.2}$$

$$= -\frac{2}{\kappa^2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}\left(\Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\rho} - \Gamma^\rho{}_{\mu\nu}\Gamma^\lambda{}_{\rho\lambda}\right) - 2\partial_\mu\phi\partial^\mu\phi\right]\ , \tag{5.3}$$

where ϕ is the dilaton, a real scalar field. We used the, already introduced, notion and strategy of section 2.3. Note that the axion is omitted in all calculations. This is due to the spinless nature of our choice of worldlines and that axions can only appear as quantum correction, at the order we are interested in, which are not contributing to the classical limit. A worldline action for a massive point particle (pm) with dilaton charge reads

$$\tilde{S}_{\text{pm}} = -\int d\tau L_{pm} = -m\int d\tau\, e^\phi\sqrt{g_{\mu\nu}u^\mu u^\nu}, \tag{5.4}$$

where we have defined the particle's velocity $u^\mu = \dot{x}^\mu$ and ϕ, $g_{\mu\nu}$ take values on the worldline $x^\mu(\tau)$. Moreover, the action enjoys a reparametrization symmetry in the worldline parameter τ. At this point we would like to make our lives easier by removing the square root. One way to do so is to switch to canonical variables, i.e. we define new canonical momenta

$$p_\mu = \frac{\partial L_{\text{pm}}}{\partial u^\mu} = m\frac{e^\phi\, u_\mu}{\sqrt{g_{\mu\nu}u^\mu u^\nu}}, \qquad \Rightarrow \qquad m^2 = e^{-2\phi}\, p^2, \tag{5.5}$$

compute the "canonical", H_{can}, and "Dirac" Hamiltonian, H_D,

$$\begin{aligned} H_{\text{can}} &= p_\mu u^\mu - L_{pm} = 0\ , \\ H_D &= H_{\text{can}} + \lambda\left(e^{-2\phi}\, p^2 - m^2\right)\ , \end{aligned} \tag{5.6}$$

[1] Implicitly adding the GHY boundary term for mathematical consistency.

where we have introduced a Lagrange multiplier $\lambda(\tau)$ (note that $\lambda(\tau)d\tau$ is reparametrization invariant) . Finally, we obtain the new, so called "first order formalism", action

$$S_{\text{pm}} = -\int d\tau \, (p_\mu u^\mu - H_D) \tag{5.7}$$

$$= -\int d\tau \left(p_\mu u^\mu - \lambda \left[e^{-2\phi} g^{\mu\nu} p_\mu p_\nu - m^2\right]\right). \tag{5.8}$$

Thus the full system of two worldlines interacting in classical dilaton-gravity (cdg) is given by

$$S_{\text{cdg}} = S_{\text{dg}} + S_{\text{gf}} + S_{\text{pm}} + \tilde{S}_{\text{pm}}, \tag{5.9}$$

where we add a gauge fixing part S_{gf} which will be specified below. The two independent worldlines are kept notationally distinct by placing a tilde atop all variables of one worldline. Now we introduce gravitons through the graviton expansion (2.53). As expected, we arrive at a huge, rather unpractical, analytic expression. Luckily, as was shown in [117], it is possible to perform simple field redefinitions and appropriate gauge fixings such that a remarkable simplification takes place. In our case, we also choose to add various total derivative terms. Our goal is transform the resulting expressions such that we can find as much YM structure as possible in the Feynman rules. We begin by choosing the gauge fixing term up to $\mathcal{O}(\kappa^2)$

$$S_{\text{gf}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} f^\mu f_\mu, \tag{5.10}$$

$$\begin{aligned} f^\mu = \Gamma^\mu{}_{\nu\sigma} \, g^{\nu\sigma} + \frac{\kappa^2}{2}\Big[&-\frac{1}{4}(\partial_\kappa \, h^{\kappa\lambda}) \, h_\lambda{}^\mu - \frac{1}{4}(\partial^\mu \, h^{\kappa\lambda}) h_{\kappa\lambda} + (\partial^\kappa \, h^{\mu\lambda}) h_{\kappa\lambda} \\ &+ \frac{3}{16}(\partial^\mu \, h^\kappa{}_\kappa) \, h^\lambda{}_\lambda - \frac{3}{8}(\partial^\kappa \, h^\mu{}_\kappa) \, h^\lambda{}_\lambda - \frac{3}{8}(\partial^\lambda \, h^\kappa{}_\kappa) \, h^\mu{}_\lambda \Big] \,, \end{aligned} \tag{5.11}$$

where the first term is understood to be graviton expanded (2.53), i.e. it is De-Donder gauge at leading order. Next, we perform the field redefinitions

$$\begin{aligned} h_{\mu\nu} \quad &\rightarrow \quad h_{\mu\nu} - \eta_{\mu\nu}\left(\frac{1}{2} h^\mu{}_\mu + 2\phi\right) \\ &+ \kappa\Big(-\frac{1}{2} h_{\mu\nu} \, h^\rho{}_\rho + \frac{1}{8} \eta_{\mu\nu} \, h^\rho{}_\rho \, h^\sigma{}_\sigma + \frac{1}{2} h_{\mu\rho} \, h^\rho{}_\nu - 2\,\phi \, h_{\mu\nu} + 2\,\phi^2 \, \eta_{\mu\nu} + \phi \, h_{\mu\nu} \, h^\rho{}_\rho\Big), \end{aligned} \tag{5.12}$$

$$\phi \quad \rightarrow \quad \phi + \frac{1}{4} h^\mu{}_\mu \,. \tag{5.13}$$

Finally, we add the total derivatives

$$\begin{aligned} 0 = S_{\text{TD}} = \int d^4x \Big[&\partial_\mu(\, (\partial_\nu \, h^{\mu\kappa}) \, h^\nu{}_\kappa) - \partial_\mu \, (h^{\mu\nu} \, (\partial_\kappa \, h^\kappa{}_\nu)) \\ &+ \kappa \, (\frac{1}{4} \partial_\mu (h^{\mu\nu} (\partial_\nu \, h^{\sigma\nu}) \, h_{\sigma\nu}) - \frac{1}{4} \partial_\mu \, ((\partial_\nu \, h^{\mu\lambda}) h_{\lambda\kappa} \, h^{\kappa\nu}) \\ &- \frac{1}{4} \partial_\mu \, (h^{\mu\nu} \, (\partial_\lambda \, h_{\nu\rho}) h^{\rho\lambda}) + \frac{1}{4} \partial_\mu \, (h^\mu{}_\nu \, h^{\nu\lambda} \, (\partial_\sigma \, h_\lambda{}^\sigma)))\Big] \,. \end{aligned} \tag{5.14}$$

Doing so, we end up with the following simplifications. First, the dilaton decouples from the worldline of the massive point particle up to $\mathcal{O}(\kappa^2)$

$$S_{\rm pm} = -\int d\tau \left(p_\mu u^\mu - \lambda \left[\left(\eta^{\mu\nu} - \kappa h^{\mu\nu} + \frac{\kappa^2}{2} h^\mu{}_\rho h^{\rho\nu} \right) p_\mu p_\nu - m^2 \right] \right) + \mathcal{O}(\kappa^3). \tag{5.15}$$

Second, the three graviton interaction term takes a very elegant form pointing directly at a double copy structure, i.e. the full resulting action up to $\mathcal{O}(\kappa)$ reads

$$S_{\rm dg} + S_{\rm gf} + S_{TD} \tag{5.16}$$

$$= \int d^4x \Big[\frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{\kappa}{4} \Big(h_{\mu\nu}\, \partial^\mu\, \partial^\nu\, h_{\rho\sigma}\, h^{\rho\sigma} + 2\, h_{\mu\nu}\, \partial^\sigma\, h^\mu{}_\rho\, \partial^\nu\, h^\rho{}_\sigma \tag{5.17}$$

$$- h_{\mu\nu}\, \partial^\sigma\, h_\rho{}^\mu\, \partial^\rho\, h^\nu{}_\sigma - h_{\rho\sigma}\, \partial^\rho\, h_{\mu\nu}\, \partial^\sigma\, h^{\mu\nu} - \partial_\rho\, \partial_\sigma\, h_{\mu\nu}\, h^{\rho\mu}\, h^{\sigma\nu} \Big) \Big] + \mathcal{O}(\kappa^2, \phi)$$

$$= \int d^4x \left[\frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{\kappa}{4 \cdot 3!} V_{123}^{\mu\alpha\gamma} V_{123}^{\nu\beta\delta} h_{1\mu\nu} h_{2\alpha\beta} h_{3\gamma\delta} \right] + \mathcal{O}(\kappa^2, \phi) \ , \tag{5.18}$$

where $V_{123}^{\mu\alpha\gamma}$ is given by

$$V_{123}^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2} \left(\partial_1^{\mu_3} - \partial_2^{\mu_3} \right) + \text{cyclic} \ , \tag{5.19}$$

i.e. it is the dynamical part of the YM three gluon vertex in position space. The indices 1, 2, 3, on $h_{\mu\nu}$ highlight the action of the partial derivatives in $V_{123}^{\mu_1\mu_2\mu_3}$. Note that we are neglecting all terms $\mathcal{O}(\kappa^2, \phi)$ as they cannot contribute to the classical effective action.

Now we turn to the Feynman rules. Graphically, we will denote the worldlines as solid lines and gravitons as double-wiggly lines. The propagator is given by

$$\langle h_{\mu\nu}(x)\, h_{\rho\sigma}(y) \rangle_0 \ = -\frac{\hbar}{i}\, \eta_{\mu(\rho}\eta_{\sigma)\nu}\, D(x-y) \ , \tag{5.20}$$

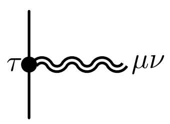
where we have defined $D(x-y)$ such that it satisfies

$$\Box D(x-y) = -\delta(x-y) \ , \tag{5.21}$$

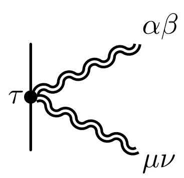
and the standard symmetrization prescription applies

$$\eta_{\mu(\rho}\eta_{\sigma)\nu} = \frac{1}{2} \left(\eta_{\mu\rho}\, \eta_{\sigma\nu} + \eta_{\mu\sigma}\, \eta_{\rho\nu} \right) \ . \tag{5.22}$$

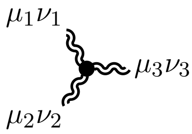
The vertices read



$$= -\frac{i}{\hbar}\, \kappa\, \lambda(\tau)\, p^\mu(\tau)\, p^\nu(\tau), \tag{5.23}$$



$$= \frac{i}{\hbar}\, \kappa^2\, \lambda(\tau)\, p^{(\mu}(\tau)\, \eta^{\nu)(\alpha}\, p^{\beta)}(\tau), \tag{5.24}$$



$$= \frac{i}{\hbar}\,\frac{\kappa}{4}\,V_{123}^{\alpha_1\alpha_2\alpha_3}\,V_{123}^{\beta_1\beta_2\beta_3}\prod_{i=1}^{3} I_{\alpha_i\beta_i}{}^{\mu_i\nu_i}, \tag{5.25}$$

with the tensor $I_{\mu\nu\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta})$.

5.1.2 Worldlines interacting through Yang-Mills

The question of defining an action for a classical colored point charge (pc) moving along a worldline $x^\mu(\tau)$ has been addressed in [118]. The action reads

$$\tilde{S}_{\rm pc} = -\int d\tau L_{\rm pc} = -\int d\tau \left(m\sqrt{u^2} - \psi^\dagger i u^\mu D_\mu \psi\right) \tag{5.26}$$

$$= -\int d\tau \left(m\sqrt{u^2} - i\psi^\dagger\dot{\psi} - g u^\mu A_\mu^a c^a\right), \tag{5.27}$$

where we have introduced an auxiliary field $\psi(\tau)$, in the fundamental representation of the YM gauge group, carrying the color degrees of freedom along the worldline and

$$c^a := \psi^\dagger T^a \psi\,, \tag{5.28}$$

has been defined as the classical color charge of the point particle. We can already see that the double copy procedure might run into trouble since we lack a two gluon coupling to the worldline which is present on the dilaton-gravity side. However, we can resolve the tension by switching to the first order formalism as we have done in the case of dilaton-gravity. We define the canonical momenta

$$p_\mu = \frac{\partial L_{\rm pc}}{\partial u^\mu} = m\frac{u_\mu}{\sqrt{u^2}} - gA_\mu^a c^a, \qquad \Rightarrow \qquad m^2 = (p+gAc)^2, \tag{5.29}$$

Next, we compute the "Dirac" Hamiltonian H_D from the "canonical" Hamiltonian $H_{\rm can}$ by adding the mass-shell constraint using a Lagrange multiplier $\lambda(\tau)$, i.e.

$$H_D = H_{\rm can} + \lambda\left[(p+gAc)^2 - m^2\right], \qquad H_{\rm can} = p_\mu u^\mu - L_{pc} \overset{(5.29)}{=} i\psi^\dagger\dot{\psi}, \tag{5.30}$$

$$\tag{5.31}$$

and thus we arrive at the first order formalism action

$$\begin{aligned} S_{\rm pc} &= -\int d\tau (p_\mu u^\mu - H_{\rm pc}) \\ &= -\int d\tau \left(p_\mu u^\mu - i\psi^\dagger\dot{\psi} - \lambda\left[p^2 + 2gp_\mu A_a^\mu c^a + g^2 A_\mu^b c^b A_a^\mu c^a - m^2\right]\right). \end{aligned} \tag{5.32}$$

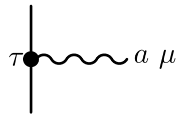
The action of the full system of two color charged point particles interacting in YM reads

$$S_{\mathrm{cYM}} = S_{\mathrm{YM}} + S_{\mathrm{gf}} + S_{\mathrm{pc}} + \tilde{S}_{\mathrm{pc}} \tag{5.33}$$

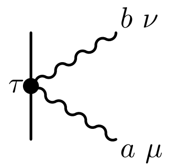
where the subscript cYM stands for classical Yang-Mills and a gauge fixing term S_{gf} was added. As before, we separate the notation of the two worldline by a tilde. The position space propagator reads

$$\langle A^a_\mu(x)\, A^b_\nu(y)\rangle_0 \;=\; \frac{\hbar}{i}\,\eta_{\mu\nu}\,\delta_{ab}\,D(x-y)\ . \tag{5.34}$$

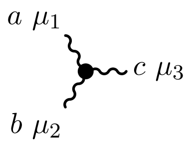
The Vertices read



$$= \frac{i}{\hbar}\,2g\,\lambda(\tau)\,p^\mu(\tau)\,c^a(\tau), \tag{5.35}$$



$$= \frac{i}{\hbar}2g^2\,\lambda(\tau)\,c^a(\tau)\,c^b(\tau)\,\eta^{\mu\nu}, \tag{5.36}$$



$$= -\frac{i}{\hbar}g\,f^{abc}\,V^{\mu_1\mu_2\mu_3}_{123}\ , \tag{5.37}$$

where $V^{\mu_1\mu_2\mu_3}_{123}$ is given by (5.19).

5.2 The Effective action

We are now turning to the computation of the classical effective action for both systems. We will start by computing the classical effective action of two color charged massive particles interacting in YM followed by the same computation where the two particles are not color charged anymore but interact in dilaton-gravity. After that we will compare both results and deduce the correct color-to-kinematics replacement rule. We will compare the emerging prescription to the standard double copy procedure between scattering amplitudes in YM and dilaton-gravity as we have discussed in sec. 2.6.

5.2.1 The classical effective action of two colored point particles

We now compute the classical effective action for two color charged massive point particles moving on their worldlines up to $\mathcal{O}(g^4)$, $S_{\mathrm{eff,cYM}}$, by integrating out the gauge boson A^a_μ, i.e.

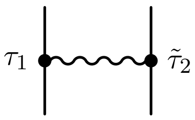
$$e^{\frac{i}{\hbar}S_{\mathrm{eff,cYM}}} = e^{\frac{i}{\hbar}S_{\mathrm{pc,free}}}\,\mathcal{M}_{\mathrm{cYM}} = \mathcal{C}_{\mathrm{YM}}\int \mathcal{D}A\,e^{\frac{i}{\hbar}S_{\mathrm{cYM}}}\ , \tag{5.38}$$

where the normalization $\mathcal{C}_{\mathrm{YM}}$ is chosen such that $\mathcal{M}_{\mathrm{YM}} \to 1$ for $g \to 0$ and $S_{\mathrm{pc,free}} = S_{\mathrm{pc}} + \tilde{S}_{\mathrm{pc}}$ for $g \to 0$. We will use the notation

$$c_i := c(\tau_i) \quad , \quad p_i := p(\tau_i) \quad , \quad D_{ij} := D(x(\tau_i) - x(\tau_j)) \quad , \quad d\hat{\tau}_{1\ldots n} := \prod_{i=1}^{n} d\tau_i\,\lambda(\tau_i)\,. \tag{5.39}$$

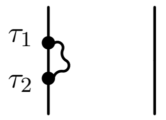
Additionally, we will use the tilde notation for the "right" worldline in our diagrams.

At $\mathcal{O}(g^2)$ we only have one non-vanishing, independent analytic structure dictated by the graph

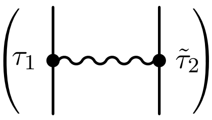


$$= 4g^2 \frac{i}{\hbar} \int d\hat{\tau}_{1\tilde{2}} (c_1 \cdot \tilde{c}_2) \, (p_1 \cdot \tilde{p}_2) \;\; D_{1\tilde{2}} \, . \tag{5.40}$$

Other graphs can be directly obtained through a replacement of tilded and untilded variables while keeping track of possible prefactors emerging from a discrepancy of orders in the Taylor expansion of the worldline action exponential. More explicitly, another graph at this order can be obtained in the following way

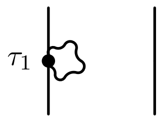


$$= \frac{1}{2} \left(\; \right) \Bigg|_{\tilde{c}_2 \to c_2 \quad , \quad \tilde{p}_2 \to p_2} \, , \tag{5.41}$$



where the factor $\frac{1}{2}$ originates from the second order expansion of the "left" worldline action in the left graph. We will call such operations redressing a graph and we will not explicitly include such graphs in our computations.

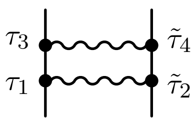
Another class of graphs at this order is trivially vanishing in dimensional regularization



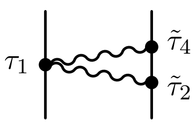
$$\propto \int d\tau_1 D_{11} = 0 \quad \text{in dimreg} \, . \tag{5.42}$$

As a consequence we will neglect all graphs involving such a bubble.

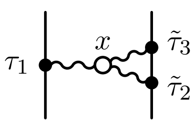
At $\mathcal{O}(g^4)$ we have the graphs



$$= 8\, g^4 \left(\frac{i}{\hbar} \right)^2 \int d\hat{\tau}_{1\tilde{2}3\tilde{4}} \, (c_1 \cdot \tilde{c}_2) \, (c_3 \cdot \tilde{c}_4) \, (p_1 \cdot \tilde{p}_2) \, (p_3 \cdot \tilde{p}_4) \, D_{1\tilde{2}} \, D_{3\tilde{4}} \, , \tag{5.43}$$

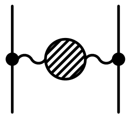


$$= 4g^4 \, \frac{i}{\hbar} \int d\hat{\tau}_{1\tilde{2}\tilde{4}} \, (c_1 \cdot \tilde{c}_2) \, (c_1 \cdot \tilde{c}_4) \, (\tilde{p}_2 \cdot \tilde{p}_4) \, D_{1\tilde{2}} \, D_{1\tilde{4}} \, , \tag{5.44}$$



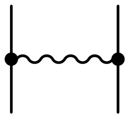
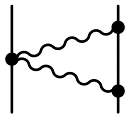
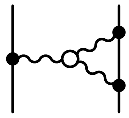
$$= \; -4\, g^4 \, \frac{i}{\hbar} \int d\hat{\tau}_{1\tilde{2}\tilde{3}} \, f^{abc} \, c_1^a \tilde{c}_2^b \tilde{c}_3^c \, V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}} \, p_{1\mu} \, \tilde{p}_{2\nu} \, \tilde{p}_{3\rho} \, G_{1\tilde{2}\tilde{3}} \, , \tag{5.45}$$

where $G_{1\tilde{2}\tilde{3}} = \int d^4x D_{1x} D_{\tilde{2}x} D_{\tilde{3}x}$. All other graphs at this order are either redressed versions of the above three graphs or quantum corrections, i.e. graphs which are $\mathcal{O}(\hbar)$, which we drop due to our focus on the classical part of the effective action. More explicitly, we neglect all graphs of the form

 , (5.46)

where only loops are present in the bubble. Taking the logarithm of (5.38) up to $\mathcal{O}(g^4)$ we obtain

$$S_{\text{eff,cYM}} = S_{\text{free,pc}} + \frac{\hbar}{i}\log \mathcal{M}_{\text{cYM}}$$

$= S_{\text{free,pc}} +$  $+$  $+$  $+ \,(\text{redressed})$

$$\begin{aligned} &= S_{\text{free,pc}} + 4g^2 \int d\hat{\tau}_{1\tilde{2}} (c_1 \cdot \tilde{c}_2)\,(p_1 \cdot \tilde{p}_2)\, D_{1\tilde{2}} \\ &\quad + 4g^4 \int d\hat{\tau}_{1\tilde{2}\tilde{4}}\, (c_1 \cdot \tilde{c}_2)\,(c_1 \cdot \tilde{c}_4)\,(\tilde{p}_2 \cdot \tilde{p}_4)\, D_{1\tilde{2}}\, D_{1\tilde{4}} \\ &\quad - 4g^4 \int d\hat{\tau}_{1\tilde{2}\tilde{3}}\, f^{abc}\, c_1^a \tilde{c}_2^b \tilde{c}_3^c\, V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}}\, p_{1\mu}\, \tilde{p}_{2\nu}\, \tilde{p}_{3\rho}\, G_{1\tilde{2}\tilde{3}} + (\text{redressed}) + \mathcal{O}(\hbar)\ , \end{aligned} \tag{5.47}$$

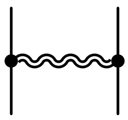
with $V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}}$ from (5.19).

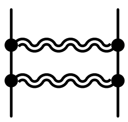
5.2.2 The classical effective action of two massive point particles

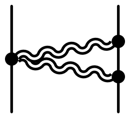
Similar to the Yang-Mills case in the previous section, we compute $S_{\text{eff,dg}}$ by integrating out the graviton and scalar fields,

$$e^{\frac{i}{\hbar} S_{\text{eff,dg}}} = e^{\frac{i}{\hbar} S_{\text{pm,free}}}\, \mathcal{M}_{\text{dg}} = \mathcal{C}_{\text{dg}} \int \mathcal{D}h\, \mathcal{D}\phi\, e^{\frac{i}{\hbar} S_{\text{cdg}}}\ , \tag{5.48}$$

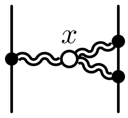
where we again defined $\mathcal{C}_{\text{dg}}$ such that $\mathcal{M}_{\text{dg}} = 1$ for $\kappa \to 0$ and $S_{\text{pm,free}} = S_{\text{pm}} + \tilde{S}_{\text{pm}}$ for $\kappa \to 0$. As before, we will neglect all loop level graphs and also keep all redressed graphs implicit. Up to $\mathcal{O}(\kappa^4)$ we have the relevant graphs

τ_1  $\tilde{\tau}_2 = -\dfrac{i\kappa^2}{\hbar} \displaystyle\int d\hat{\tau}_{1\tilde{2}}\, (p_1 \cdot \tilde{p}_2)^2 D_{1\tilde{2}}\,,$ (5.49)

τ_3, τ_1  $\tilde{\tau}_4$, $\tilde{\tau}_2$ $= \dfrac{i^2\kappa^4}{2\hbar^2} \displaystyle\int d\hat{\tau}_{1\tilde{2}\tilde{3}4}\, (p_1 \cdot \tilde{p}_2)^2 (p_4 \cdot \tilde{p}_3)^2 D_{1\tilde{2}} D_{4\tilde{3}}\,,$ (5.50)

τ_1  $\tilde{\tau}_4$ $\tilde{\tau}_2$

$$= \frac{i\kappa^4}{2\hbar}\int d\hat{\tau}_{1\tilde{2}\tilde{3}}\,(p_1\cdot\tilde{p}_2)(p_1\cdot\tilde{p}_3)(\tilde{p}_2\cdot\tilde{p}_3)D_{1\tilde{2}}D_{1\tilde{3}}\ , \tag{5.51}$$

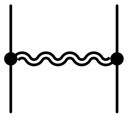
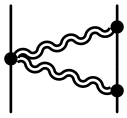
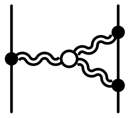
τ_1 

$\tilde{\tau}_3$ $\tilde{\tau}_2$

$$= \frac{i\kappa^4}{8\hbar}\int d\hat{\tau}_{1\tilde{2}\tilde{3}}\,\left(V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}}\,p_{1\mu}\tilde{p}_{2\nu}\tilde{p}_{3\rho}\right)^2 G_{1\tilde{2}\tilde{3}}\,. \tag{5.52}$$

Therefore, the action reads

$$S_{\text{eff,dg}} = S_{\text{free,dg}} + \frac{\hbar}{i}\log\mathcal{M}_{dg}$$

$= S_{\text{free,dg}} +$  $+$  $+$  $+$ (redressed)

$$\begin{aligned}
&= S_{\text{free,dg}} - \kappa^2\int d\hat{\tau}_{1\tilde{2}}\,(p_1\cdot\tilde{p}_2)^2 D_{1\tilde{2}} \\
&\quad + \frac{\kappa^4}{2}\int d\hat{\tau}_{1\tilde{2}\tilde{3}}\,(p_1\cdot\tilde{p}_2)(p_1\cdot\tilde{p}_3)(\tilde{p}_2\cdot\tilde{p}_3)D_{1\tilde{2}}D_{1\tilde{3}} \\
&\quad + \frac{\kappa^4}{8}\int d\hat{\tau}_{1\tilde{2}\tilde{3}}\,\left(V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}}\,p_{1\mu}\tilde{p}_{2\nu}\tilde{p}_{3\rho}\right)^2 G_{1\tilde{2}\tilde{3}} + \text{(redressed)} + \mathcal{O}(\hbar)\ .
\end{aligned} \tag{5.53}$$

5.3 The double copy procedure

In this section we want to check whether our previous results are consistent with known double copy principles. We will investigate similarities and differences between our computation of YM and dilaton-gravity effective actions and scattering amplitudes computations where the double copy is very well studied and controlled.

5.3.1 Failue of the double copy?

We begin with a consistency check of an already proposed replacement rule by W. Goldberger and A. Ridgway. In their initial paper concerning the classical double copy for generic orbits [33] they find the following replacement rules

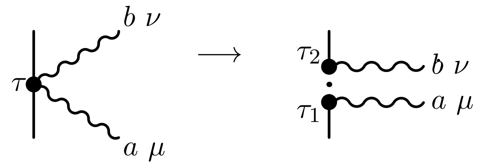
$$\begin{aligned}
g &\to \frac{\kappa}{2}\ , \\
c_i^a(\tau) &\to i\,p_i^\mu(\tau)\ , \\
f^{abc}\,c_i^a\,c_j^b\,c_k^c &\to -\frac{1}{2}\,V^{\mu\nu\rho}_{i,j,k}\,p_{\mu,i}\,p_{\nu,j}\,p_{\rho,k}\ .
\end{aligned} \tag{5.54}$$

It is very easy to see that this double copy fails in our results. The mismatch can be traced back to a factor of 2 in graph (5.44). In some sense, this is a welcome observation since (5.54) breaks with the traditional double copy approach known from a scattering amplitude context where full color factors are replaced by full kinematic factors. This is actually exactly what Goldberger and Ridgway did since, due to the lower order computation, those have been the only color structures present. In our calculation new color structures appear and we have to think of a new prescription suitable for us. The main guiding principle will be the traditional double copy prescription. In particular, if we compare our computation to a scattering amplitude one 2.6, we notice that we have not used crucial properties such as on-shell conditions for the gluons and gravitons, trivalent representations and color-kinematics duality (2.100). It is worthwhile to discuss the relevance of those properties in our computation. We directly see that on-shell conditions cannot be used since all gluons and gravitons only appear as virtual particles. Color-kinematics duality cannot appear du to the low order of our computation and also due to the observation that on-shell conditions are usually of utmost importance for its validity. Thus, two out of three ingredients are not applicable in our computation. Therefore, if we want to find a double copy prescription which works for us, we should be using the trivalent representation as a guiding principle. In an amplitudes calculation, we know that using a trivalent representation means to decompose the four gluon vertex into a linear combination of s-, t- and u-channel-like terms and add the resulting contributions to respective diagrams containing the s-, t- and u-channel. Furthermore, the relevant color structures for the double copy are identified through a trivalent representation and thus help identifying the color dual numerators. In our computation we do not include the four gluon vertex, thus we may ask whether we have any reason to think about a trivalent representation at all. Indeed, even though the amplitudes point of view does not clearly push us in this direction, we can quickly see that we have a four vertex like building block (5.36) where two gluons couple to the same point on a classical colored worldline. We stress that it is far from obvious to identify (5.36) as a four vertex since is not a vertex in a QFT sense because worldlines are not asymptotic states and thus cannot be seen as external particles. Moreover, worldlines cannot propagate which makes it far from straightforward inventing a new decomposition in this context, i.e. it is not obvious what the s-,t- and u-channel like structures are.

Having identified the problem of a non-functioning double copy introduced by W. Goldberger and A. Ridgway [33], we are now turning towards a resolution through an appropriate implementation of a trivalent representation.

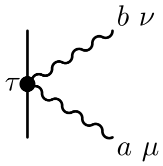
5.3.2 The trivalent representation and the double copy prescription of the classical effective action for two colored point particles

As we have discussed in the previous section, we want to use a notion of a trivalent representation mirroring its double copy importance in a scattering amplitude context. Our goal is to decompose the two-gluon-worldline vertex (5.36) by pulling apart the contact, i.e.



(5.55)

We do this by inserting a delta function, $\delta(\tau_1 - \tau_2)$, localized by a worldline integral



$$= \frac{i}{\hbar} 2g^2 \, \lambda(\tau)\, c^a(\tau)\, c^b(\tau)\, \eta^{\mu\nu} = \frac{i}{\hbar} 2g^2 \int d\tau_2 \, \lambda(\tau_1)\, c^a(\tau_1)\, c^b(\tau_2)\, \eta^{\mu\nu}\, \delta(\tau_1 - \tau_2) \ . \tag{5.56}$$

Using this decomposition allows us to combine the graph (5.44) with (5.43) into one expression since the color factors can be made identical. Thus, for the double copy to work, we propose that at any order in g, $\mathcal{M}_{\text{cYM}}$ must be brought into the analytic form

$$\mathcal{M}_{\text{cYM}} = 1 + \sum_{n=1}^{\infty} (2g)^{2n} \sum_{I \in \Gamma_n} \left(\frac{i}{\hbar}\right)^{x_I} \int \prod_{i_I} d\hat{\tau}_{i_I} \int d^{4l_I} x \, \frac{C_I \, N_I}{S_I \, D_I} \ . \tag{5.57}$$

We have defined the quantity Γ_n as the set of N^{n-1}LO graphs in the above defined trivalent representation, C_I denotes the color factor associated with graph I, i.e. functions of the $c(\tau)$'s, N_I are the associated kinematic numerator factors and D_I are the spacetime propagators appearing in graph I. The object x_I is defined as the number of vertices (bulk and worldline) minus the number of propagators in the trivalent diagram and l_I is the number of bulk vertices. Finally, S_I is the symmetry factor of graph I in the trivalent representation. Doing so, we claim that $\mathcal{M}_{\text{dg}}$ takes the form

$$\mathcal{M}_{\text{dg}} = 1 + \sum_{n=1}^{\infty} (i\kappa)^{2n} \sum_{I \in \Gamma_n} \left(\frac{i}{\hbar}\right)^{x_I} \int \prod_{i_I} d\hat{\tau}_{i_I} \int d^{4l_I} x \, \frac{N_I \, N_I}{S_I \, D_I} \ . \tag{5.58}$$

It is crucial to note that the double copy is defined for $\mathcal{M}$ and not the final effective action, i.e. it has to be performed before taking the logarithm.

5.3.3 The double copy of the classical effective action for two colored point particles

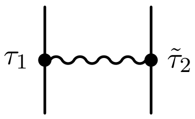
We are now aiming at actually performing the double copy according to (5.57). We begin by setting up the trivalent representation of $\mathcal{M}_{\text{cYM}}$. The independent color structures are given by

$$\mathcal{O}(g^2): \qquad (c \cdot \tilde{c}) \ ,$$

$$\mathcal{O}(g^4): \qquad (c \cdot \tilde{c})^2 \ , \ f^{abc} c_a \tilde{c}_b \tilde{c}_c \ .$$

These color structures define the C_I in (5.57) and all other color structures are their redressed versions. The relevant graphs in our double copy prescription are:

- The H-graph at $\mathcal{O}(g^2)$

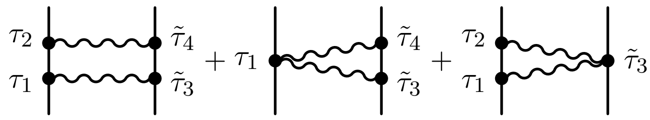


$$H : \tau_1 \ \ \tilde{\tau}_2 \quad = (2g)^2 \left(\frac{i}{\hbar}\right) \int d\hat{\tau}_{1\tilde{2}} \, \frac{C_H \, N_H}{S_H \, D_H} \ , \tag{5.59}$$

where

$$C_H = (c_1 \cdot \tilde{c}_2) \quad , \quad D_H^{-1} = D_{1\tilde{2}} \quad , \quad S_H = 1 \quad , \quad N_H = (p_1 \cdot \tilde{p}_2) \ . \tag{5.60}$$

- At $\mathcal{O}(g^4)$ we are looking at the V-graph



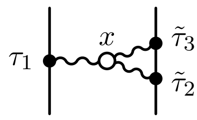
$$V : \quad = (2g)^4 \left(\frac{i}{\hbar}\right)^2 \int d\hat{\tau}_{12\tilde{3}\tilde{4}} \, \frac{C_V \, N_V}{S_V \, D_V} \ , \tag{5.61}$$

where

$$\begin{aligned} &D_V^{-1} = D_{1\tilde{3}} D_{2\tilde{4}} \qquad , \qquad C_V = (c_1 \cdot \tilde{c}_3)(c_2 \cdot \tilde{c}_4) \qquad , \qquad S_V = 2 \ , \\ &N_V = (p_1 \cdot \tilde{p}_3)\,(p_2 \cdot \tilde{p}_4) \ + \frac{\hbar}{2i}\,\frac{\delta(\tau_1 - \tau_2)}{\lambda_2}\,(\tilde{p}_3 \cdot \tilde{p}_4) + \frac{\hbar}{2i}\,\frac{\delta(\tilde{\tau}_3 - \tilde{\tau}_4)}{\tilde{\lambda}_4}\,(p_1 \cdot p_2) \ . \end{aligned} \tag{5.62}$$

Here, we used our proposed prescription of a trivalent graph representation (5.56). Note that the symmetry factor strictly belongs to the first (double ladder) graph and differs from the latter two contact term graphs. This is exactly the way in which the symmetry factor is defined in the double copy (amplitudes and ours) prescription.

- The Y-graph



$$Y : \quad = (2g)^4 \left(\frac{i}{\hbar}\right) \int d\hat{\tau}_{1\tilde{2}\tilde{3}} \, \frac{C_Y \, N_Y}{S_Y \, D_Y} \ , \tag{5.63}$$

where

$$\begin{aligned} &C_Y = f^{abc}\, c_1^a \tilde{c}_2^b \tilde{c}_3^c \qquad , \qquad D_Y^{-1} = G_{1\tilde{2}\tilde{3}} \qquad , \qquad S_Y = 2 \ , \\ &N_Y = -\frac{1}{2} V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}} \, p_{1\mu} \, \tilde{p}_{2\nu} \, \tilde{p}_{3\rho} \ . \end{aligned} \tag{5.64}$$

In the exact same fashion we can organize all redressed versions of above graphs. Performing the double copy according to (5.58) yields

$$\begin{aligned} \bar{\mathcal{M}}_{\rm dg} = 1 - \kappa^2 \, \frac{i}{\hbar} \int d\hat{\tau}_H \, \frac{N_H \, N_H}{S_H \, D_H} + \kappa^4 \left(\frac{i}{\hbar}\right)^2 \int d\hat{\tau}_V \, \frac{N_V \, N_V}{S_V \, D_V} \\ + \kappa^4 \, \frac{i}{\hbar} \int d\hat{\tau}_Y \, \frac{N_Y \, N_Y}{S_Y \, D_Y} + (\text{redressed}) \ , \end{aligned} \tag{5.65}$$

where the bar notation for $\bar{\mathcal{M}}_{\rm dg}$ keeps track of the use of our double copy prescription. Above expression has one subtlety. The product $N_V\, N_V$ contains terms $\delta(0)$ which is rather problematic on first glance. Nevertheless, such terms are $\mathcal{O}(\hbar^0)$ and thus they fall into the class of quantum corrections to $S_{\rm eff,dg}$ and we boldly neglect them.

For the effective action of $S_{\rm eff,cYM}$ we replace $\mathcal{M}_{\rm cYM}$ by $\bar{\mathcal{M}}_{\rm dg}$ in

$$S_{\rm eff,cYM} = S_{\rm free,pc} + \frac{\hbar}{i}\log \mathcal{M}_{\rm cYM}\ ,$$

and obtain

$$\begin{aligned}
\bar{S}_{\rm eff,dg} = S_{\rm free,pc} + \frac{\hbar}{i}\log \tilde{\mathcal{M}}_{\rm dg} &= S_{\rm free,pc} - \kappa^2 \int d\hat{\tau}_{1\tilde{2}}\, (p_1\cdot\tilde{p}_2)^2 D_{1\tilde{2}} \\
&+ \frac{\kappa^4}{2}\int d\hat{\tau}_{1\tilde{2}\tilde{3}}\, (p_1\cdot\tilde{p}_2)(p_1\cdot\tilde{p}_3)(\tilde{p}_2\cdot\tilde{p}_3) D_{1\tilde{2}} D_{1\tilde{3}} \\
&+ \frac{\kappa^4}{8}\int d\hat{\tau}_{1\tilde{2}\tilde{3}}\, \left(V^{\mu\nu\rho}_{1\tilde{2}\tilde{3}}\, p_{1\mu}\tilde{p}_{2\nu}\tilde{p}_{3\rho}\right)^2 G_{1\tilde{2}\tilde{3}} + (\text{redressed}) + \mathcal{O}(\hbar),
\end{aligned} \tag{5.66}$$

which, indeed, leads to the remarkable identification (5.53)

$$\bar{S}_{\rm eff,dg} = S_{\rm eff,dg} + \int d\tau\, i(\psi^\dagger\dot{\psi} + \tilde{\psi}^\dagger\dot{\tilde{\psi}})\ . \tag{5.67}$$

where we have used $S_{\rm free,pc} = S_{\rm free,dg} + \int d\tau\, i(\psi^\dagger\dot{\psi} + \tilde{\psi}^\dagger\dot{\tilde{\psi}})$. The last term can even be dropped trivially due to the decoupling of ψ's dynamics. We conclude that our double-copy prescription (5.58) maps the classical effective action of two classically colored massive point particles interacting in YM to the effective action of two classically dilaton charged massive point particles interacting in dilaton-gravity up to next-to-leading order.

Chapter 6

Summary and Outlook

Motivated by the existence of the double copy prescription [11–13], we explored various connections between gluons and gravitons with the goal of uncovering, identifying and understanding new aspects of this mysterious, remarkable and completely unobvious property of gauge theory.

In the first project contained in chapter 3, we performed an in-depth analysis of a connection between a pair of adjacent collinear gluons and a graviton (2.103) as proposed by Stieberger and Taylor [38], which we called ST-one . We proved this connection through an involved study of collinear gluon limits in the CHY formalism which allowed us to also prove the universality of the sub-leading collinear gluon limit. Furthermore, we were able to constrain a sub-leading collinear gluon theorem. Moreover, a rich and elegant structure emerged, connecting ST-one to a deviation from BCJ relations (2.19) and a gauge invariance violation in the fused gluon. In particular, we have learned that a deviation from BCJ relations is necessary for the existence of EYM amplitudes according to (3.107) and the violation of gauge invariance in the fused gluon manifests gauge invariance of the EYM amplitude. Still, it is highly unobvious and unintuitive that fundamental properties of existence and gauge invariance of EYM amplitudes are connected to the sub-leading collinear gluon limit. It is unclear how those, highly unintuitive, observations can be understood from a non technical point of view, i.e. whether a symmetry explanation can be given. Clearly this is one possible direction for future research even though no straightforward next step is identifiable. Further questions in this direction are very hard to identify due to the mysterious nature of ST-one. Arguably the best follow up question concern the factorization of the sub-leading collinear gluon limit, i.e. solving the constraints (3.114) and (3.116). It might also be worthwhile to study cases where more adjacent collinear gluon pairs are present. Or it might be of interest to get an in-depth analytic understanding of the neglected non-degenerate solutions of the scattering equations in the collinear limit. Finally, it might be fruitful to repeat the same computation for the case of mixed helicity collinear gluons where an analogous ST-one formula is expected due to the trivial cancellation of problematic terms in the sum-over solutions. Maybe a Feynman diagrammatic approach for low-point amplitudes can give further insight into the mysterious structures obtained from ST-one.

In the second project contained in chapter 4, we have analyzed another connection proposed by Stieberger and Taylor [39] in which a weighted linear combination of YM amplitudes produces an EYM amplitude. We called this connection ST-three (2.108) and we have been able to achieve our goal of proving its universality in the presence of massive fundamental matter consisting of quarks and scalars. We began by gaining arguments on why ST-three should be universal from a first principle point of view followed by hard evidence based on the CHY formula. This gave us enough motivation to proceed with our, almost fully unsupported by any other research, claim (4.1). We then performed the proof through a Feynman diagrammatic analysis which revealed two novel observations, namely that the number of graviton insertions is equal to the number of gluon insertion into a full sQCD tree level amplitude and that a simple color-to-kinematics replacement rule exists which directly maps sQCD amplitudes to EsQCD amplitudes. Two more important consequences have been revealed from this project. First, it was shown that there exist relations between gluons and gravitons outside of fully massless and adjoint particle scattering scenarios. This implies that if there existed an all-unifying underlying principle behind such relations, it should be a relation between single particles and not only between massless amplitudes. The second consequence concerns the gap between ST-three and the double copy procedure in terms of their execution. While the double copy procedure strictly relates full amplitudes through a color-to-kinematics replacement rule based on color dual numerators identified by color kinematics duality (2.100), ST-relations work at the level of color ordered amplitudes with no involvement of color at all. Our Jacobi relations satisfying replacement rule (4.46) thus sets both types of gluon/graviton relations on equal footing from a conceptual point of view, linking ST-three to a form of the double copy. Further directions include the repetition of our analysis for more than one graviton insertion. This will quickly prove to be highly complicated due to the involvement of the three graviton vertex but it is very likely to produce a positive result due to the existence of such formulae in the case of EYM amplitudes where full control is already established [43, 45–47] based on gauge invariance through BCJ relations. This is exactly the same situation with which we started building up evidence for universality. Furthermore, higher graviton cases in the EYM case can be recursively obtained from the single graviton case through gauge invariance alone which strongly implies its universality from a first principles point of view.

In the final project contained in chapter 5, we showed that the double copy logic can be extended even further into unknown territory. In particular, we extended the work initiated by Goldberger and Ridgway [31, 33] to a classical system of two point particles, without any emission of radiation, using the effective action as the relevant quantity for the double copy. We have found a replacement rule which correctly maps between the effective action of a massive classically colored worldline system interacting in YM and the effective action of a massive classical worldline system interacting in dilaton gravity up to next-to-leading order. We have encountered direct problems in the replacement rule proposed by Goldberger and Ridgway which we have been able to resolve by introducing a new notion of a trivalent representation on the gauge theory side. Two crucial ingredients have been used along the way. In both systems we have used the first order formalism for simplification on the gravity side and for the introduction of higher vertices on the gauge theory side. On the gravity side we have also used a set of field redefinitions, gauge fixings and total derivatives and found a remarkable structure almost enabling a double copy at the level of Feynman rules. We have also used a rather opaque, but successful, assumption in our computation, arguing that the terms proportional to $\delta(0)$, which are automatically introduced by our double copy prescription, are neglectable due to their quantum nature, i.e. being of $\mathcal{O}(\hbar)$ in the effective action. Strictly speaking, we should not have been allowed to do so and our justification is legitimized by our result. It is unclear whether it is possible to find a different prescription such that inconsistencies of this type are avoided. It is important to point out that we have achieved even slightly more than a novel double copy prescription for effective actions. We observe that our double copy prescription does not use on-shell conditions and gauge invariance. Both ingredients are usually at the core of the double copy which makes our computation more intriguing from a hidden symmetry point of view. This brings up the question of whether there is an accidental reason explaining the lack of necessity for on-shell conditions and gauge invariance while somehow enforcing them in a scattering amplitude setting. One possible further direction of research includes a repetition of the same computation at next-to-next-to-leading order where more double copy properties are expected to emerge, i.e. color dual numerators. Unfortunately, a very recent consistency check [119] has revealed a significant mismatch leading to the conclusion that a next-to-next-to-leading order double copy does not exist. Other possible research could be done by giving both worldlines spin and repeat the computation. First work has already been done by W. Goldberger, J. Li and S. G. Prabhu [32]. Here, novel structures are expected to emerge due to the need for an axion on the gravity side. Identifying methods which can remove the dilaton from final results is also a highly relevant and interesting direction for further research. In the case of classical limits from scattering amplitudes precise elimination procedures have been recently found by Z. Bern et al. [120] where the dilaton degrees of freedom are removed through simple on-shell conditions. This is obviously not possible in our computation. Finally, it might be worthwhile to study our computation in a scenario where the auxiliary field ψ is also integrated out on the gauge theory side. This would introduce a propagator on the worldline which may lead to novel analytic insights into the classical double copy of effective actions.

It remains to see whether any of the results obtained in this thesis can be of value for both the identification of the YM kinematic algebra and a consistent, UV complete theory of quantum gravity.

Bibliography

[1] C. N. Yang and R. L. Mills, *"Conservation of Isotopic Spin and Isotopic Gauge Invariance"*, Phys. Rev. 96, 191 (1954), `https://link.aps.org/doi/10.1103/PhysRev.96.191`.

[2] P. W. Higgs, *"Broken Symmetries and the Masses of Gauge Bosons"*, Phys. Rev. Lett. 13, 508 (1964), `https://link.aps.org/doi/10.1103/PhysRevLett.13.508`.

[3] Virgo, LIGO Scientific Collaboration, B. P. Abbott et al., *"Observation of Gravitational Waves from a Binary Black Hole Merger"*, Phys. Rev. Lett. 116, 061102 (2016), `arxiv:1602.03837`.

[4] ATLAS Collaboration, G. Aad et al., *"Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC"*, Phys. Lett. B716, 1 (2012), `arxiv:1207.7214`.

[5] CMS Collaboration, S. Chatrchyan et al., *"Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC"*, Phys. Lett. B716, 30 (2012), `arxiv:1207.7235`.

[6] R. J. Adler, B. Casey and O. C. Jacob, *"Vacuum catastrophe: An elementary exposition of the cosmological constant problem"*, American Journal of Physics - AMER J PHYS 63, 620 (1995).

[7] K. C. Freeman, *"On the disks of spiral and SO Galaxies"*, Astrophys. J. 160, 811 (1970).

[8] B. S. DeWitt, *"Quantum Theory of Gravity. 2. The Manifestly Covariant Theory"*, Phys. Rev. 162, 1195 (1967), [,298(1967)].

[9] J. F. Donoghue, M. M. Ivanov and A. Shkerin, *"EPFL Lectures on General Relativity as a Quantum Field Theory"*, `arxiv:1702.00319`.

[10] B. S. DeWitt, *"Quantum Theory of Gravity. 3. Applications of the Covariant Theory"*, Phys. Rev. 162, 1239 (1967), [,307(1967)].

[11] Z. Bern, J. J. M. Carrasco and H. Johansson, *"New Relations for Gauge-Theory Amplitudes"*, Phys. Rev. D78, 085011 (2008), `arxiv:0805.3993`.

[12] Z. Bern, J. J. M. Carrasco and H. Johansson, *"Perturbative Quantum Gravity as a Double Copy of Gauge Theory"*, Phys. Rev. Lett. 105, 061602 (2010), `arxiv:1004.0476`.

[13] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *"Gravity as the Square of Gauge Theory"*, Phys. Rev. D82, 065003 (2010), `arxiv:1004.0693`.

[14] Z. Bern, C. Boucher-Veronneau and H. Johansson, *"$\mathcal{N} \geq 4$ Supergravity Amplitudes from Gauge Theory at One Loop"*, Phys. Rev. D84, 105035 (2011), `arxiv:1107.1935`.

[15] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, *"Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes"*, Phys. Rev. D85, 105014 (2012), `arxiv:1201.5366`.

[16] J. J. M. Carrasco, M. Chiodaroli, M. Gunaydin and R. Roiban, *"One-loop four-point amplitudes in pure and matter-coupled $\mathcal{N} \leq 4$ supergravity"*, JHEP 1303, 056 (2013), arxiv:1212.1146.

[17] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang and J. Nohle, *"Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops"*, Phys. Rev. D92, 045041 (2015), arxiv:1303.6605.

[18] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *"Scattering amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein supergravity"*, JHEP 1501, 081 (2015), arxiv:1408.0764.

[19] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *"Complete construction of magical, symmetric and homogeneous N=2 supergravities as double copies of gauge theories"*, Phys. Rev. Lett. 117, 011603 (2016), arxiv:1512.09130.

[20] G. Mogull and D. O'Connell, *"Overcoming Obstacles to Colour-Kinematics Duality at Two Loops"*, JHEP 1512, 135 (2015), arxiv:1511.06652.

[21] S. He, R. Monteiro and O. Schlotterer, *"String-inspired BCJ numerators for one-loop MHV amplitudes"*, JHEP 1601, 171 (2016), arxiv:1507.06288.

[22] H. Johansson and A. Ochirov, *"Pure Gravities via Color-Kinematics Duality for Fundamental Matter"*, JHEP 1511, 046 (2015), arxiv:1407.4772.

[23] A. Anastasiou, L. Borsten, M. J. Duff, M. J. Hughes, A. Marrani, S. Nagy and M. Zoccali, *"Twin supergravities from Yang-Mills theory squared"*, Phys. Rev. D96, 026013 (2017), arxiv:1610.07192.

[24] A. Anastasiou, L. Borsten, M. J. Duff, A. Marrani, S. Nagy and M. Zoccali, *"Are all supergravity theories Yang-Mills squared?"*, arxiv:1707.03234.

[25] C. Cheung, C.-H. Shen and C. Wen, *"Unifying Relations for Scattering Amplitudes"*, JHEP 1802, 095 (2018), arxiv:1705.03025.

[26] H. Johansson and A. Ochirov, *"Color-Kinematics Duality for QCD Amplitudes"*, JHEP 1601, 170 (2016), arxiv:1507.00332.

[27] H. Johansson, G. Mogull and F. Teng, *"Unraveling conformal gravity amplitudes"*, JHEP 1809, 080 (2018), arxiv:1806.05124.

[28] Z. Bern, J. J. Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R. Roiban and M. Zeng, *"Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops"*, arxiv:1804.09311.

[29] R. Monteiro and D. O'Connell, *"The Kinematic Algebra From the Self-Dual Sector"*, JHEP 1107, 007 (2011), arxiv:1105.2565.

[30] C. Cheung and C.-H. Shen, *"Symmetry for Flavor-Kinematics Duality from an Action"*, Phys. Rev. Lett. 118, 121601 (2017), arxiv:1612.00868.

[31] W. D. Goldberger and A. K. Ridgway, *"Radiation and the classical double copy for color charges"*, Phys. Rev. D95, 125010 (2017), arxiv:1611.03493.

[32] W. D. Goldberger, J. Li and S. G. Prabhu, *"Spinning particles, axion radiation, and the classical double copy"*, arxiv:1712.09250.

[33] W. D. Goldberger and A. K. Ridgway, *"Bound states and the classical double copy"*, arxiv:1711.09493.

[34] R. Monteiro, D. O'Connell and C. D. White, *"Black holes and the double copy"*, JHEP 1412, 056 (2014), arxiv:1410.0239.

[35] A. Luna, R. Monteiro, D. O'Connell and C. D. White, *"The classical double copy for Taub NUT spacetime"*, Phys. Lett. B750, 272 (2015), arxiv:1507.01869.

[36] D. Chester, *"Radiative double copy for Einstein-Yang-Mills theory"*, Phys. Rev. D97, 084025 (2018), arxiv:1712.08684.

[37] C.-H. Shen, *"Gravitational Radiation from Color-Kinematics Duality"*, arxiv:1806.07388.

[38] S. Stieberger and T. R. Taylor, *"Subleading terms in the collinear limit of Yang?Mills amplitudes"*, Phys. Lett. B750, 587 (2015), arxiv:1508.01116.

[39] S. Stieberger and T. R. Taylor, *"New relations for Einstein–Yang–Mills amplitudes"*, Nucl. Phys. B913, 151 (2016), arxiv:1606.09616.

[40] S. He, O. Schlotterer and Y. Zhang, *"New BCJ representations for one-loop amplitudes in gauge theories and gravity"*, Nucl. Phys. B930, 328 (2018), arxiv:1706.00640.

[41] S. He and O. Schlotterer, *"New Relations for Gauge-Theory and Gravity Amplitudes at Loop Level"*, Phys. Rev. Lett. 118, 161601 (2017), arxiv:1612.00417.

[42] J. Faller and J. Plefka, *"Positive helicity Einstein-Yang-Mills amplitudes from the double copy method"*, Phys. Rev. D99, 046008 (2019), arxiv:1812.04053.

[43] D. Nandan, J. Plefka, O. Schlotterer and C. Wen, *"Einstein-Yang-Mills from pure Yang-Mills amplitudes"*, JHEP 1610, 070 (2016), arxiv:1607.05701.

[44] M. Chiodaroli, M. Gunaydin, H. Johansson and R. Roiban, *"Explicit Formulae for Yang-Mills-Einstein Amplitudes from the Double Copy"*, JHEP 1707, 002 (2017), arxiv:1703.00421.

[45] F. Teng and B. Feng, *"Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame"*, JHEP 1705, 075 (2017), arxiv:1703.01269.

[46] Y.-J. Du, B. Feng and F. Teng, *"Expansion of All Multitrace Tree Level EYM Amplitudes"*, JHEP 1712, 038 (2017), arxiv:1708.04514.

[47] C.-H. Fu, Y.-J. Du, R. Huang and B. Feng, *"Expansion of Einstein-Yang-Mills Amplitude"*, JHEP 1709, 021 (2017), arxiv:1702.08158.

[48] F. Cachazo, S. He and E. Y. Yuan, *"Scattering of Massless Particles in Arbitrary Dimensions"*, Phys. Rev. Lett. 113, 171601 (2014), arxiv:1307.2199.

[49] F. Cachazo, S. He and E. Y. Yuan, *"Scattering of Massless Particles: Scalars, Gluons and Gravitons"*, JHEP 1407, 033 (2014), arxiv:1309.0885.

[50] F. Cachazo, S. He and E. Y. Yuan, *"Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations"*, JHEP 1501, 121 (2015), arxiv:1409.8256.

[51] D. Nandan, J. Plefka and G. Travaglini, *"All rational one-loop Einstein-Yang-Mills amplitudes at four points"*, arxiv:1803.08497.

[52] T. Kinoshita, *"Mass singularities of Feynman amplitudes"*, J. Math. Phys. 3, 650 (1962).

[53] T. D. Lee and M. Nauenberg, *"Degenerate Systems and Mass Singularities"*, Phys. Rev. 133, B1549 (1964), https://link.aps.org/doi/10.1103/PhysRev.133.B1549.

[54] S. Weinberg, *"The Quantum theory of fields. Vol. 1: Foundations"*, Cambridge Univ. Press , 1 (1995).

[55] S. Weinberg, *"The quantum theory of fields. Vol. 2: Modern applications"*, Cambridge University Press (2013).

[56] M. E. Peskin and D. V. Schroeder, *"An Introduction to quantum field theory"*, Addison-Wesley (1995), Reading, USA.

[57] M. D. Schwartz, *"Quantum Field Theory and the Standard Model"*, Cambridge University Press (2014).

[58] L. J. Dixon, *"Calculating scattering amplitudes efficiently"*, hep-ph/9601359.

[59] H. Elvang and Y.-t. Huang, *"Scattering Amplitudes in Gauge Theory and Gravity"*, Cambridge University Press , (2015), arxiv:1308.1697.

[60] J. M. Henn and J. C. Plefka, *"Scattering Amplitudes in Gauge Theories"*, Lect. Notes Phys. 883, 1 (2014).

[61] S. J. Parke and T. R. Taylor, *"Amplitude for n-Gluon Scattering"*, Phys. Rev. Lett. 56, 2459 (1986), https://link.aps.org/doi/10.1103/PhysRevLett.56.2459.

[62] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov and J. Trnka, *"Grassmannian Geometry of Scattering Amplitudes"*, Cambridge University Press (2016).

[63] H. Elvang and Y.-t. Huang, *"Scattering Amplitudes in Gauge Theory and Gravity"*, Cambridge University Press (2015).

[64] R. Kleiss and H. Kuijf, *"Multi - Gluon Cross-sections and Five Jet Production at Hadron Colliders"*, Nucl. Phys. B312, 616 (1989).

[65] V. Del Duca, L. J. Dixon and F. Maltoni, *"New color decompositions for gauge amplitudes at tree and loop level"*, Nucl. Phys. B571, 51 (2000), hep-ph/9910563.

[66] B. Feng, R. Huang and Y. Jia, *"Gauge Amplitude Identities by On-shell Recursion Relation in S-matrix Program"*, Phys. Lett. B695, 350 (2011), arxiv:1004.3417.

[67] T. Melia, *"Proof of a new colour decomposition for QCD amplitudes"*, JHEP 1512, 107 (2015), arxiv:1509.03297.

[68] A. A. Slavnov, *"Ward Identities in Gauge Theories"*, Theor. Math. Phys. 10, 99 (1972), [Teor. Mat. Fiz.10,153(1972)].

[69] J. Taylor, *"Ward identities and charge renormalization of the Yang-Mills field"*, Nuclear Physics B 33, 436 (1971), http://www.sciencedirect.com/science/article/pii/0550321371902975.

[70] T. Melia, *"Getting more flavor out of one-flavor QCD"*, Phys. Rev. D89, 074012 (2014), arxiv:1312.0599.

[71] L. de la Cruz, A. Kniss and S. Weinzierl, *"Proof of the fundamental BCJ relations for QCD amplitudes"*, JHEP 1509, 197 (2015), arxiv:1508.01432.

[72] R. W. Brown and S. G. Naculich, *"Color-factor symmetry and BCJ relations for QCD amplitudes"*, JHEP 1611, 060 (2016), arxiv:1608.05291.

[73] G. W. Gibbons and S. W. Hawking, *"Action Integrals and Partition Functions in Quantum Gravity"*, Phys. Rev. D15, 2752 (1977).

[74] E. Dyer and K. Hinterbichler, *"Boundary Terms, Variational Principles and Higher Derivative Modified Gravity"*, Phys. Rev. D79, 024028 (2009), arxiv:0809.4033.

[75] R. M. Wald, *"General Relativity"*, Chicago Univ. Pr. (1984), Chicago, USA.

[76] G. 't Hooft and M. J. G. Veltman, *"One loop divergencies in the theory of gravitation"*, Ann. Inst. H. Poincare Phys. Theor. A20, 69 (1974).

[77] T. Schuster, *"Lee-Wick Gauge Theory and Effective Quantum Gravity"*, Diploma thesis, 2008, Humboldt University Berlin, availabe at http://qft.physik.hu-berlin.de.

[78] S. Weinberg, *"Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass"*, Phys. Rev. 135, B1049 (1964).

[79] F. Low, *"Bremsstrahlung of very low-energy quanta in elementary particle collisions"*, Phys. Rev. 110, 974 (1958).

[80] T. Burnett and N. M. Kroll, *"Extension of the low soft photon theorem"*, Phys. Rev. Lett. 20, 86 (1968).

[81] E. Casali, *"Soft sub-leading divergences in Yang-Mills amplitudes"*, JHEP 1408, 077 (2014), arxiv:1404.5551.

[82] F. Cachazo and A. Strominger, *"Evidence for a New Soft Graviton Theorem"*, arxiv:1404.4091.

[83] A. Strominger, *"On BMS Invariance of Gravitational Scattering"*, arxiv:1312.2229.

[84] T. He, V. Lysov, P. Mitra and A. Strominger, *"BMS supertranslations and Weinberg's soft graviton theorem"*, arxiv:1401.7026.

[85] T. He, P. Mitra and A. Strominger, *"2D Kac-Moody Symmetry of 4D Yang-Mills Theory"*, arxiv:1503.02663.

[86] R. H. Boels and W. Wormsbecher, *"Spontaneously broken conformal invariance in observables"*, arxiv:1507.08162.

[87] P. Di Vecchia, R. Marotta, M. Mojaza and J. Nohle, *"New soft theorems for the gravity dilaton and the Nambu-Goldstone dilaton at subsubleading order"*, Phys. Rev. D93, 085015 (2016), arxiv:1512.03316.

[88] N. Arkani-Hamed, L. Rodina and J. Trnka, *"Locality and Unitarity of Scattering Amplitudes from Singularities and Gauge Invariance"*, Phys. Rev. Lett. 120, 231602 (2018), arxiv:1612.02797.

[89] Z. Bern, L. J. Dixon and D. A. Kosower, *"Two-loop g —¿ gg splitting amplitudes in QCD"*, JHEP 0408, 012 (2004), hep-ph/0404293.

[90] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, *"One loop n point helicity amplitudes in (selfdual) gravity"*, Phys. Lett. B444, 273 (1998), hep-th/9809160.

[91] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, *"Multileg one loop gravity amplitudes from gauge theory"*, Nucl. Phys. B546, 423 (1999), hep-th/9811140.

[92] T. Azevedo, M. Chiodaroli, H. Johansson and O. Schlotterer, *"Heterotic and bosonic string amplitudes via field theory"*, JHEP 1810, 012 (2018), arxiv:1803.05452.

[93] H. Kawai, D. C. Lewellen and S. H. H. Tye, *"A Relation Between Tree Amplitudes of Closed and Open Strings"*, Nucl. Phys. B269, 1 (1986).

[94] C. R. Mafra, O. Schlotterer and S. Stieberger, *"Explicit BCJ Numerators from Pure Spinors"*, JHEP 1107, 092 (2011), arxiv:1104.5224.

[95] F. Cachazo, S. He and E. Y. Yuan, *"Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM"*, JHEP 1507, 149 (2015), arxiv:1412.3479.

[96] F. Cachazo, S. He and E. Y. Yuan, *"New Double Soft Emission Theorems"*, arxiv:1503.04816.

[97] L. Dolan and P. Goddard, *"Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension"*, JHEP 1405, 010 (2014), arxiv:1311.5200.

[98] D. B. Fairlie, *"A Coding of Real Null Four-Momenta into World-Sheet Co-ordinates"*, Adv. Math. Phys. 2009, 284689 (2009), arxiv:0805.2263.

[99] D. J. Gross and P. F. Mende, *"String Theory Beyond the Planck Scale"*, Nucl. Phys. B303, 407 (1988).

[100] E. Witten, *"Parity invariance for strings in twistor space"*, Adv. Theor. Math. Phys. 8, 779 (2004), hep-th/0403199.

[101] P. Caputa and S. Hirano, *"Observations on Open and Closed String Scattering Amplitudes at High Energies"*, JHEP 1202, 111 (2012), arxiv:1108.2381.

[102] P. Caputa, *"Lightlike contours with fermions"*, Phys. Lett. B716, 475 (2012), arxiv:1205.6369.

[103] Y. Makeenko and P. Olesen, *"The QCD scattering amplitude from area behaved Wilson loops"*, Phys. Lett. B709, 285 (2012), arxiv:1111.5606.

[104] F. Cachazo, *"Fundamental BCJ Relation in N=4 SYM From The Connected Formulation"*, arxiv:1206.5970.

[105] E. Casali, *"Worldsheet methods for perturbative quantum field theory"*, https://www.repository.cam.ac.uk/handle/1810/265833.

[106] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily, P. H. Damgaard and B. Feng, *"Integration Rules for Loop Scattering Equations"*, JHEP 1511, 080 (2015), arxiv:1508.03627.

[107] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily and P. H. Damgaard, *"Scattering Equations and Feynman Diagrams"*, JHEP 1509, 136 (2015), arxiv:1507.00997.

[108] C. Baadsgaard, N. E. J. Bjerrum-Bohr, J. L. Bourjaily and P. H. Damgaard, *"Integration Rules for Scattering Equations"*, JHEP 1509, 129 (2015), arxiv:1506.06137.

[109] D. Nandan, J. Plefka and W. Wormsbecher, *"Collinear limits beyond the leading order from the scattering equations"*, JHEP 1702, 038 (2017), arxiv:1608.04730.

[110] L. Dolan and P. Goddard, *"The Polynomial Form of the Scattering Equations"*, JHEP 1407, 029 (2014), arxiv:1402.7374.

[111] J. Plefka and W. Wormsbecher, *"New relations for graviton-matter amplitudes"*, Phys. Rev. D98, 026011 (2018), arxiv:1804.09651.

[112] S. G. Naculich, *"CHY representations for gauge theory and gravity amplitudes with up to three massive particles"*, JHEP 1505, 050 (2015), arxiv:1501.03500.

[113] S. G. Naculich, *"Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles"*, JHEP 1409, 029 (2014), arxiv:1407.7836.

[114] J. Plefka, J. Steinhoff and W. Wormsbecher, *"Effective action of dilaton gravity as the classical double copy of Yang-Mills theory"*, Phys. Rev. D99, 024021 (2019), arxiv:1807.09859.

[115] W. D. Goldberger, S. G. Prabhu and J. O. Thompson, *"Classical gluon and graviton radiation from the bi-adjoint scalar double copy"*, Phys. Rev. D96, 065009 (2017), arxiv:1705.09263.

[116] A. Luna, I. Nicholson, D. O'Connell and C. D. White, *"Inelastic Black Hole Scattering from Charged Scalar Amplitudes"*, JHEP 1803, 044 (2018), arxiv:1711.03901.

[117] Z. Bern and A. K. Grant, *"Perturbative gravity from QCD amplitudes"*, Phys. Lett. B457, 23 (1999), hep-th/9904026.

[118] A. P. Balachandran, P. Salomonson, B.-S. Skagerstam and J.-O. Winnberg, *"Classical Description of Particle Interacting with Nonabelian Gauge Field"*, Phys. Rev. D15, 2308 (1977).

[119] J. Plefka, C. Shi, J. Steinhoff and T. Wang, *"Breakdown of the classical double copy for the effective action of dilaton-gravity at NNLO"*, arxiv:1906.05875.

[120] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon and M. Zeng, *"Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order"*, Phys. Rev. Lett. 122, 201603 (2019), arxiv:1901.04424.